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Logical Quantization of Topos Theory

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Traditionally set theory lies at the hub of all mathematics in the sense that every branch of mathematics, ranging from algebraic geometry to functional analysis, is to be considered as developed within some formal system of set theory. Recently topos theory, which is a natural generalization of set theory, has provided an alternative foundation of mathematics, not to say the foundation of mathematics. With these considerations in mind, we quantize topos theory logically along the lines of our previous papers. The paper culminates in the quantum treatment of classifying toposes.

INTRODUCTION

Since Cantor, *set theory* has traditionally been considered to be a formal vessel in which every mathematical activity takes place. Therefore any upheaval in set theory gravely affects all mathematics, as was demonstrated by the discovery of paradoxes in Cantorian set theory during the early years of this century, which was followed by the creation of a new branch of mathematics responsible for foundations of mathematics, namely, *metamathematics*. Cohen's (1966) epoch-making work established the independence of the continuum hypothesis from Zermelo–Fraenkel set theory and finally led to the creation of another new branch of mathematics called *Boolean-valued analysis* (Takeuti, 1978). Boolean-valued analysis enables us to approach some traditional areas of mathematics such as functional analysis from a new and perspicacious viewpoint. By way of example, Ozawa (1984) succeeded in settling negatively Kaplansky's (1953, p. 843, footnote) long-baffling problem on the uniqueness of the direct sum decomposition of type I AW^*-algebras into homogeneous algebras.

What are now called *Grothendieck toposes* were introduced by Grothendieck and his collaborators in the 1960s so as to provide an adequate frame-

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work for the development of *étale cohomology* (Artin *et al.*, 1972). Indeed the power of the general machinery of French school was shown by Deligne's (1974) resolution of the Weil conjectures, which are the mod-p analogue of the Riemann hypothesis. Independently of Grothendieck's school, Lawvere (1964) embarked upon an intrepid enterprise of providing a chastely categorical foundation of all mathematics. No sooner had he heard of Grothendieck toposes than he realized that they admit the basic operations of set theory. During 1969–1970, Lawvere, in collaboration with Tierney, wrote out a large part of the basic theory of elementary toposes, which was to be regarded as the dawn of *topos theory*, a principal branch of category theory claiming to be a categorical axiomatization of *sheaf theory*.

Since the 1970s Foulis and Randall (1972; Randall and Foulis, 1973) and their collaborators have enunciated and developed a theory of manuals of operations as a formal framework for the foundations of all empirical sciences, including quantum theory. In their literature an *operation* is thought of as a set of possible outcomes and a *manual of operations* is established as a family of partially overlapping operations subject to mild constraints. In Nishimura (1993) we introduced two major viewpoints. The first is that an operation is reckoned not as a set of possible outcomes, but as a complete Boolean algebra of observable events; thereby the notion of a manual of operations is replaced by that of a *manual of Boolean locales*, a subcategory of the dual category of the category of complete Boolean algebras and their complete Boolean homomorphisms subject to a few reasonable conditions. The second is that, since each complete Boolean algebra $\mathbf{B}$ enjoys its Boolean-valued set theory $V^{(\mathbf{B})}$ and each complete Boolean homomorphism $\varphi: \mathbf{B} \to \mathbf{B}'$ induces a geometric morphism from the topos of sets and functions within $V^{(\mathbf{B}')}$ to that within $V^{(\mathbf{B})}$, a manual of Boolean locales yields a family of Boolean localic toposes interconnected by geometric morphisms as its *empirical set theory*. Since Nishimura (1995c) we have been engaged in quantizing various mathematical structures along these lines under the slogan of *logical quantization* (Nishimura, 1996a,b).

Recently topos theory has provided an alternative foundation of mathematics, permeating through the sanctuary of traditional set theory. We now feel obliged to give a fuller and more coherent treatment of logical quantization of topos theory than the makeshift ones in our previous papers (Nishimura, 1995c, n.d.-a,b); this is the principal concern of this paper. The paper culminates in the quantum treatment of classifying toposes of rings and local rings in the last section, which can be generalized easily to classifying toposes of geometric theories. Here a ring always means a commutative ring with unity and a homomorphism of rings is required to preserve unities. For the geometric definition of a local ring, the reader is referred to MacLane and Moerdijk (1992, Chapter VIII, §6).

1. CATEGORY THEORY

1.1. Elements of Category Theory

In this paper a category always means a category whose morphisms form a set. Therefore a *category* $\mathbf{C}$ is a 6-tuple (Ob $\mathbf{C}$, Mor $\mathbf{C}$, $d_{\mathbf{C}}$, $r_{\mathbf{C}}$, $\mathrm{id}_{\mathbf{C}}$, $\circ_{\mathbf{C}}$), where:

(1.1.1) Ob $\mathbf{C}$ is a set whose elements are called *objects*.
(1.1.2) Mor $\mathbf{C}$ is a set whose elements are called *morphisms*.
(1.1.3) $d_{\mathbf{C}}$ and $r_{\mathbf{C}}$ are functions from Mor $\mathbf{C}$ to Ob $\mathbf{C}$.
(1.1.4) $\mathrm{id}_{\mathbf{C}}$ is a function from Ob $\mathbf{C}$ to Mor $\mathbf{C}$ such that

$$d_{\mathbf{C}}(\mathrm{id}_{\mathbf{C}}(x)) = r_{\mathbf{C}}(\mathrm{id}_{\mathbf{C}}(x)) = x \qquad \text{for any} \quad x \in \mathrm{Ob}\,\mathbf{C}$$

(1.1.5) $\circ_{\mathbf{C}}$ is a function from

$$\mathrm{Mor}\,\mathbf{C} \times_{\mathrm{Ob}\,\mathbf{C}} \mathrm{Mor}\,\mathbf{C} = \{(g, f) \in \mathrm{Mor}\,\mathbf{C} \times \mathrm{Mor}\,\mathbf{C} \mid d_{\mathbf{C}}(g) = r_{\mathbf{C}}(f)\}$$

to Mor $\mathbf{C}$ [the value of $\circ_{\mathbf{C}}$ at (g, f) is usually denoted by $g \circ_{\mathbf{C}} f$] such that $d_{\mathbf{C}}(g \circ_{\mathbf{C}} f) = d_{\mathbf{C}}(f)$ and $r_{\mathbf{C}}(g \circ_{\mathbf{C}} f) = r_{\mathbf{C}}(g)$ for any $(g, f) \in \mathrm{Mor}\,\mathbf{C} \times_{\mathrm{Ob}\,\mathbf{C}} \mathrm{Mor}\,\mathbf{C}$.

(1.1.6) $\circ_{\mathbf{C}}$ is required to satisfy the associative law, and $\mathrm{id}_{\mathbf{C}}(x)$ is required to play a role of two-sided identity for each $x \in \mathrm{Ob}\,\mathbf{C}$.

Unless confusion may arise, the subscript $\mathbf{C}$ in $d_{\mathbf{C}}$, $r_{\mathbf{C}}$, $\mathrm{id}_{\mathbf{C}}$, and $\circ_{\mathbf{C}}$ is omitted, and $\mathrm{id}_{\mathbf{C}}(x)$ is usually written id_x.

We assume that the reader is well conversant with the fundamentals of category theory, for which the standard reference is MacLane (1971). In particular, the reader should feel at home with such locutions as a functor, a natural transformation, the opposite category $\mathbf{C}^{\mathrm{op}}$ of a category $\mathbf{C}$, a limit, a colimit, etc. A terminal object of a category $\mathbf{C}$, if it exists, is denoted by $1_{\mathbf{C}}$ or 1.

Given two natural transformations $\alpha: F \to G$ and $\beta: G \to H$ of functors from a category $\mathbf{C}$ to a category $\mathbf{D}$, their *vertical composite* $\beta \cdot \alpha$ is defined and is a natural transformation from F to H.

Given a functor $F: \mathbf{B} \to \mathbf{C}$, a natural transformation $\alpha: G \to G'$ between functors from $\mathbf{A}$ to $\mathbf{B}$, and a natural transformation $\beta: H \to H'$ between functors from $\mathbf{C}$ to $\mathbf{D}$, the composites $F \circ \alpha$ and $\beta \circ F$ are defined and are natural transformations from $F \circ G$ to $F \circ G'$ and from $H \circ F$ to $H' \circ F$, respectively. If $\alpha': G' \to G''$ is another natural transformation between functors from $\mathbf{A}$ to $\mathbf{B}$ and $\beta': H' \to H''$ is also another natural transformation between functors from $\mathbf{C}$ to $\mathbf{D}$, then we have the following:

(1.1.7) $F \circ (\alpha' \cdot \alpha) = (F \circ \alpha') \cdot (F \circ \alpha)$.
(1.1.8) $(\beta' \cdot \beta) \circ F = (\beta' \circ F) \cdot (\beta \circ F)$.

1.2. Universes

To dodge the famous paradoxes of set theory or to paper them over, the usage of a *universe* is a common practice in category theory. Roughly speaking, a universe is a well-behaved set closed under any standard operation of set theory. For the exact definition of a universe, the reader is referred, e.g., to MacLane (1971, Chapter I, §6), Schubert (1972, §3.2), or Borceux (1994, Vol. 1, §1.1). The existence of a universe is disputable from the standpoint of axiomatic set theory, but we assume in this paper that there are three universes V_0, V_1, and V_2 with $V_0 \in V_1 \in V_2$. Sets of V_i are called *small$_i$* ($i = 0, 1, 2$). The adjective "small$_i$" is applied to structures whose underlying sets are small$_i$. By way of example, a category $\mathbf{C}$ is called small$_i$ if Mor $\mathbf{C}$ is small$_i$, a functor of small$_i$ categories is called small$_i$, a natural transformation between small$_i$ functors is called small$_i$, and so on. A category $\mathbf{C}$ is called *small$_i$-complete* (*small$_i$-cocomplete*, resp.) if every small$_i$ diagram in $\mathbf{C}$ has a limit (a colimit, resp.) in $\mathbf{C}$. We denote by $\mathbf{Ens}_i$ the category of small$_i$ sets and small$_i$ functions. We denote by $\mathbf{Cat}_i$ the category of small$_i$ categories and small$_i$ functors. Given a small$_i$ category $\mathbf{C}$, we denote by $\mathbf{PreSh}_i(\mathbf{C})$ the category of contravariant functors from $\mathbf{C}$ to $\mathbf{Ens}_i$ and natural transformations. The Yoneda embedding of $\mathbf{C}$ into $\mathbf{PreSh}_i(\mathbf{C})$ is usually denoted by $\mathbf{y}$.

1.3. 2-Category

For the formal theory of 2-categories the reader is referred to Borceux (1994, Vol. 1, Chapter 7). Since we believe that a good example tells much about its general theory, we will explain how the set $\mathbf{CAT}_2$ of small$_2$ categories, small$_2$ functors, and small$_2$ natural transformations forms a 2-category. The set of small$_2$ categories is called the set of *objects* of $\mathbf{CAT}_2$ and is denoted by Ob $\mathbf{CAT}_2$. Small$_2$ functors and small$_2$ natural transformations are called *morphisms* (or *1-arrows*) and *2-arrows* of $\mathbf{CAT}_2$, respectively. Given two small$_2$ categories $\mathbf{A}$ and $\mathbf{B}$, it is well known that the set $\mathbf{CAT}_2(\mathbf{A}, \mathbf{B})$ of functors from $\mathbf{A}$ to $\mathbf{B}$ and natural transformations among them is a category with respect to vertical composition of natural transformations. Given three small$_2$ categories $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ and four functors $F: \mathbf{A} \to \mathbf{B}$, $G: \mathbf{A} \to \mathbf{B}$, $F': \mathbf{B} \to \mathbf{C}$, and $G': \mathbf{B} \to \mathbf{C}$, the horizontal composite $\beta \circ \alpha$ of two natural transformations $\alpha: F \to G$ and $\beta: F' \to G'$ is defined to be either side of the following well-known equality:

$$(1.3.1) \quad (\beta \circ G)\cdot(F' \circ \alpha) = (G' \circ \alpha)\cdot(\beta \circ F)$$

Given four small$_2$ categories $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, six functors $F: \mathbf{A} \to \mathbf{B}$, $G: \mathbf{A} \to \mathbf{B}$, $F': \mathbf{B} \to \mathbf{C}$, $G': \mathbf{B} \to \mathbf{C}$, $F'': \mathbf{C} \to \mathbf{D}$, and $G'': \mathbf{C} \to \mathbf{D}$, and three natural transformations $\alpha: F \to G$, $\beta: F' \to G'$, and $\gamma: F'' \to G''$, it is easy to see that

(1.3.2) $(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$

Given a small_2 functor $F: \mathbf{A} \to \mathbf{B}$, we denote by id_F the identity natural transformation from F to itself. Then, for any natural transformation $\alpha: G \to G'$ of small_2 functors $G, G': \mathbf{B} \to \mathbf{C}$ and any natural transformation $\beta: H \to H'$ of small_2 functors $H, H': \mathbf{D} \to \mathbf{A}$, we have

(1.3.3) $\alpha \circ \text{id}_F = \alpha \circ F$
(1.3.4) $\text{id}_F \circ \beta = F \circ \beta$

Given three small_2 categories **A**, **B**, and **C**, six functors $F: \mathbf{A} \to \mathbf{B}$, $G: \mathbf{A} \to \mathbf{B}$, $H: \mathbf{A} \to \mathbf{B}$, $F': \mathbf{B} \to \mathbf{C}$, $G': \mathbf{B} \to \mathbf{C}$, and $H': \mathbf{B} \to \mathbf{C}$, and four natural transformations $\alpha: F \to G$, $\beta: G \to H$, $\alpha': F' \to G'$, and $\beta': G' \to H'$, it is easy to see the following interchange law:

(1.3.5) $(\beta' \cdot \alpha') \circ (\beta \cdot \alpha) = (\beta' \circ \beta) \cdot (\alpha' \circ \alpha)$

We recapitulate:

Theorem 1.3.1. $\mathbf{CAT}_2$ is a 2-category with respect to vertical and horizontal compositions of natural transformations.

1.4. Orthogonal Category

The notion of an orthogonal category was introduced by Nishimura (1995a). In this paper we need its relativized version with respect to smallness_0. That is to say, an *orthogonal category* is a pair $(\mathbf{K}, \mathcal{O}\mathcal{S}_{\mathbf{K}})$ of a category **K** and a class $\mathcal{O}\mathcal{S}_{\mathbf{K}}$ of diagrams in **K** subject to the following conditions:

(1.4.1) The category **K** has an initial object.

(1.4.2) Every diagram in **K** is a small_0 discrete cocone $\{x \xrightarrow{f_\lambda} y\}_{\lambda \in \Lambda}$.

(1.4.3) For any small_0 family $\{x_\lambda\}_{\lambda \in \Lambda}$ of objects in **K** there exist an object y in **K** and a family $\{f_\lambda\}_{\lambda \in \Lambda}$ of morphisms $f_\lambda: x_\lambda \to y$ in **K** such that the cocone $\{x_\lambda \xrightarrow{f_\lambda} y\}_{\lambda \in \Lambda}$ lies in $\mathcal{O}\mathcal{S}_{\mathbf{K}}$.

(1.4.4) Given a small family $\{x_\lambda\}_{\lambda \in \Lambda}$ of objects in **K**, if diagrams $\{x_\lambda \xrightarrow{f_\lambda} y\}_{\lambda \in \Lambda}$ and $\{x_\lambda \xrightarrow{g_\lambda} z\}_{\lambda \in \Lambda}$ lie in $\mathcal{O}\mathcal{S}_{\mathbf{K}}$, then there exists a unique morphism $h: y \to z$ in **K** such that $g_\lambda = h \circ f_\lambda$ for each $\lambda \in \Lambda$.

(1.4.5) Given diagrams $\{y_\lambda \xrightarrow{g_\lambda} z\}_{\lambda \in \Lambda}$ and $\{x_\delta \xrightarrow{f_\delta} y_\lambda\}_{\delta \in \Delta_\lambda}$ $(\lambda \in \Lambda)$ in **K**, the diagram $\{x_\delta \xrightarrow{g_\lambda \circ f_\delta} z \mid \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ lies in $\mathcal{O}\mathcal{S}_{\mathbf{K}}$ iff the diagrams $\{y_\lambda \xrightarrow{g_\lambda} z\}_{\lambda \in \Lambda}$ and $\{x_\delta \xrightarrow{f_\delta} y_\lambda\}_{\delta \in \Delta_\lambda}$ $(\lambda \in \Lambda)$ lie in $\mathcal{O}\mathcal{S}_{\mathbf{K}}$, where the sets Δ_λ are assumed to be mutually disjoint.

(1.4.6) If a diagram $\{x_\delta \xrightarrow{f_\delta} y \mid \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ lies in $\mathcal{O}\mathcal{S}_{\mathbf{K}}$, then there exist diagrams $\{x_\delta \xrightarrow{f_\delta} z_\lambda\}_{\delta \in \Delta_\lambda}$ $(\lambda \in \Lambda)$ and $\{z_\lambda \xrightarrow{g_\lambda} y\}_{\lambda \in \Lambda}$ in $\mathcal{O}\mathcal{S}_{\mathbf{K}}$ such that $f_\delta = h_\lambda \circ g_\delta$ for any $\lambda \in \Lambda$ and any $\delta \in \Delta_\lambda$, where the sets Δ_λ are assumed to be mutually disjoint.

(1.4.7) If $\{x_\lambda \xrightarrow{f_\lambda} y\}_{\lambda \in \Lambda}$ and $\{z_\delta \xrightarrow{g_\delta} y\}_{\delta \in \Delta}$ are diagrams in $\mathbf{K}$ with z_δ being an initial object of $\mathbf{K}$ for each $\delta \in \Delta$, then the diagram $\{x_\lambda \xrightarrow{f_\lambda} y\}_{\lambda \in \Lambda}$ is in $\mathfrak{O}\mathfrak{S}_{\mathbf{K}}$ iff the diagram $\{x_\lambda \xrightarrow{f_\lambda} y\}_{\lambda \in \Lambda} \cup \{z_\delta \xrightarrow{g_\delta} y\}_{\delta \in \Delta}$ is in $\mathfrak{O}\mathfrak{S}_{\mathbf{K}}$.

(1.4.8) Given diagrams $\{x_\lambda \xrightarrow{f_\lambda} y\}_{\lambda \in \Lambda}$ and $\{x_\delta \xrightarrow{f_\delta} y\}_{\delta \in \Delta}$ in $\mathbf{K}$, with Λ and Δ being disjoint, if both the diagram $\{x_\lambda \xrightarrow{f_\lambda} y\}_{\lambda \in \Lambda}$ and the diagram $\{x_\lambda \xrightarrow{f_\lambda} y\}_{\lambda \in \Lambda} \cup \{x_\delta \xrightarrow{f_\delta} y\}_{\delta \in \Delta}$ are in $\mathfrak{O}\mathfrak{S}_{\mathbf{K}}$, then x_δ is an initial object for each $\delta \in \Delta$.

(1.4.9) If $f: x \to y$ is an isomorphism in $\mathbf{K}$, then the diagram $\{x \xrightarrow{f} y\}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathbf{K}}$.

(1.4.10) If a diagram $\{x \xrightarrow{f} y\}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathbf{K}}$, then f is an isomorphism.

(1.4.11) Given a diagram $\{x_\lambda \xrightarrow{f_\lambda} y\}_{\lambda \in \Lambda}$ in $\mathfrak{O}\mathfrak{S}_{\mathbf{K}}$, if f_{λ_1} and f_{λ_2} happen to be the same morphism for some distinct $\lambda_1, \lambda_2 \in \Lambda$ (so that $x_{\lambda_1} = x_{\lambda_2}$), then $x_{\lambda_1} = x_{\lambda_2}$ is an initial object of $\mathbf{K}$.

The diagrams in $\mathfrak{O}\mathfrak{S}_{\mathbf{K}}$ are called the *orthogonal sum diagrams* of $\mathbf{K}$. If a diagram $\{x_\lambda \to y\}_{\lambda \in \Lambda}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathbf{K}}$, then y is called an orthogonal sum of x_λ's. Unless confusion may arise, the category $\mathbf{K}$ itself is called an orthogonal category by abuse of language.

The inspiring model of an orthogonal category was the dual category $\mathbf{HLoc}_0$ of the category $\mathbf{Hil}_0$ of small_0 complex Hilbert spaces and contractive linear mappings. The dual category $\mathbf{BLoc}_0$ of the category $\mathbf{Bool}_0$ of small_0 complete Boolean algebras and complete Boolean homomorphisms is also an orthogonal category with respect to small_0 coproduct diagrams as its orthogonal sum diagrams, and will play an important role throughout this paper. The objects of $\mathbf{BLoc}_0$ are called (*small*$_0$) *Boolean locales* and are denoted by $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \ldots$. Its morphisms are denoted by $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}, \ldots$. If a Boolean locale $\mathfrak{X}$ is to be put down as an object of $\mathbf{Bool}_0$, then it is denoted by $\mathcal{P}(\mathfrak{X})$. Similarly, the morphism of $\mathbf{Bool}_0$ corresponding to $\mathfrak{f} \in \text{Mor } \mathbf{BLoc}_0$ is denoted by $\mathcal{P}(\mathfrak{f})$. Given a Boolean locale $\mathfrak{X}$ and $p \in \mathcal{P}(\mathfrak{X})$, $\mathfrak{X}_p$ denotes the Boolean locale such that $\mathcal{P}(\mathfrak{X}_p) = \{q \in \mathcal{P}(\mathfrak{X}) \mid q \leq p\}$. We denote the unit of the Boolean algebra $\mathcal{P}(\mathfrak{X})$ by $1_{\mathfrak{X}}$ or 1.

Orthogonal categories were introduced as an abstract framework upon which the formal theory of manuals initiated by Foulis and Randall (1972; Randall and Foulis, 1973) is to be developed. The full treatment of manuals within the orthogonal category $\mathbf{BLoc}_0$ by Nishimura (1993) preceded the formal introduction of orthogonal categories themselves. Using the nomenclature of Nishimura (1994a), by a *manual of Boolean locales* we will always mean a small_0 subcategory $\mathcal{M}$ of $\mathbf{BLoc}_0$ which claims to be a completely coherent rich manual within the orthogonal category $\mathbf{BLoc}_0$. An orthogonal sum diagram $\{\mathfrak{X}_\lambda \xrightarrow{\mathfrak{f}_\lambda} \mathfrak{X}\}_{\lambda \in \Lambda}$ of $\mathbf{Bloc}_0$ lying in $\mathcal{M}$ is said to be an *orthogonal $\mathcal{M}$-sum diagram* if for any orthogonal sum diagram $\{\mathfrak{X}_\lambda \xrightarrow{\mathfrak{g}_\lambda} \mathfrak{X}'\}_{\lambda \in \Lambda}$ of $\mathbf{Bloc}_0$ lying in $\mathcal{M}$, the unique morphism $\mathfrak{h}: \mathfrak{X} \to \mathfrak{X}'$ of $\mathbf{BLoc}_0$ with $\mathfrak{g}_\lambda = \mathfrak{h} \circ \mathfrak{f}_\lambda$ for all $\lambda \in \Lambda$ in condition (1.4.4) belongs to $\mathcal{M}$, in which X is denoted symbolically by $\Sigma_{\lambda \in \Lambda} \oplus_{\mathcal{M}} X_\lambda$.

2. TOPOS THEORY

For the general theory of toposes, the reader is referred to Bell (1988), Borceux (1994, Vol. 3), Goldblatt (1979), and especially MacLane and Moerdijk (1992).

2.1. The 2-Categories of Geometric Functors

Given two small$_2$ toposes $\mathbf{E}_\pm$, a *geometric functor* from $\mathbf{E}_+$ to $\mathbf{E}_-$ is a functor E: $\mathbf{E}_+ \to \mathbf{E}_-$ subject to the following conditions:

(2.1.1) E has a right adjoint E': $\mathbf{E}_- \to \mathbf{E}_+$.
(2.1.2) E is left exact.

In other words, a geometric functor from $\mathbf{E}_+$ to $\mathbf{E}_-$ is no other than the inverse image part of a geometric morphism from $\mathbf{E}_-$ to $\mathbf{E}_+$ in the standard terminology (MacLane and Moerdijk, 1992, Chapter VII, §1). We denote by $\mathbf{TOP}_2$ the 2-category of small$_2$ toposes (as objects), geometric functors among them (as morphisms), and natural transformations between parallel geometric functors (as 2-arrows), which inherits the composition of geometric functors and the two kinds of compositions of 2-arrows from the 2-category $\mathbf{CAT}_2$. The objects and morphisms of $\mathbf{TOP}_2$ constitute a category to be denoted by $\mathbf{Top}_2$.

Given two geometric functors E: $\mathbf{E}_+ \to \mathbf{E}_-$ and F: $\mathbf{F}_+ \to \mathbf{F}_-$, a *geometric conjugation** from E to F is a triple (H_-, H_+, α) of two geometric functors $H_\pm$: $\mathbf{E}_\pm \to \mathbf{F}_\pm$ and a natural transformation α: $F \circ H_+ \to H_- \circ E$. Given two geometric conjugations* (H_-, H_+, α) and (K_-, K_+, β) from E to F, a *geometric transformation** from (H_-, H_+, α) to (K_-, K_+, β) is a pair (σ_-, σ_+) of natural transformations $\sigma_\pm$: $H_\pm \to K_\pm$ subject to the following condition:

(2.1.3) $\beta \cdot (F \circ \sigma_+) = (\sigma_- \circ E) \cdot \alpha$

Given three geometric conjugations* (H_-, H_+, α), (K_-, K_+, β), and (L_-, L_+, γ) from E to F, the vertical composite of geometric transformations* (σ_-, σ_+): $(H_-, H_+, \alpha) \to (K_-, K_+, \beta)$ and (τ_-, τ_+): $(K_-, K_+, \beta) \to (L_-, L_+, \gamma)$, denoted by $(\tau_-, \tau_+) \cdot (\sigma_-, \sigma_+)$, is defined to be $(\tau_- \cdot \sigma_-, \tau_+ \cdot \sigma_+)$, for which we have:

Proposition 2.1.1. $(\tau_-, \tau_+) \cdot (\sigma_-, \sigma_+)$ is a geometric transformation* from (H_-, H_+, α) to (L_-, L_+, γ).

Proof. Since (σ_-, σ_+) and (τ_-, τ_+) are geometric transformations, they satisfy:

(2.1.4) $\beta \cdot (F \circ \sigma_+) = (\sigma_- \circ E) \cdot \alpha$.
(2.1.5) $\gamma \cdot (F \circ \tau_+) = (\tau_- \circ E) \cdot \beta$.

Therefore the desired condition

$$(2.1.6) \quad \gamma \cdot (F \circ (\tau_+ \cdot \sigma_+)) = ((\tau_- \cdot \sigma_-) \circ E) \cdot \alpha$$

follows from the following calculation:

$$
\begin{aligned}
&\gamma \cdot (F \circ (\tau_+ \cdot \sigma_+)) \\
&= \gamma \cdot ((F \circ \tau_+) \cdot (F \circ \sigma_+)) && \text{[from (1.1.7)]} \\
&= (\gamma \cdot (F \circ \tau_+)) \cdot (F \circ \sigma_+) \\
&= ((\tau_- \circ E) \cdot \beta) \cdot (F \circ \sigma_+) && \text{[from (2.1.5)]} \\
&= (\tau_- \circ E) \cdot (\beta \cdot (F \circ \sigma_+)) \\
&= (\tau_- \circ E) \cdot ((\sigma_- \circ E) \cdot \alpha) && \text{[from (2.1.4)]} \\
&= ((\tau_- \circ E) \cdot (\sigma_- \circ E)) \cdot \alpha \\
&= ((\tau_- \cdot \sigma_-) \circ E) \cdot \alpha && \text{[from (1.1.8)]} \quad \blacksquare
\end{aligned}
$$

We denote by $\mathbf{GEOM}_2^*(E, F)$ the set of geometric conjugations* from E to F and geometric transformations* among them, which is easily seen to be a category with respect to vertical composition of geometric transformations*.

Given three geometric functors $E: \mathbf{E}_+ \to \mathbf{E}_-$, $F: \mathbf{F}_+ \to \mathbf{F}_-$, and $G: \mathbf{G}_+ \to \mathbf{G}_-$, we define the composite of geometric conjugations* $(H_-, H_+, \alpha): E \to F$ and $(K_-, K_+, \beta): F \to G$, in notation $(K_-, K_+, \beta) \circ (H_-, H_+, \alpha)$, to be the geometric conjugation* $(K_- \circ H_-, K_+ \circ H_+, (K_- \circ \alpha) \cdot (\beta \circ H_+))$ from E to G. We would like to show that the set $\mathbf{Geom}_2^*$ of all geometric functors of small$_2$ toposes and geometric conjugations* among them is a category with respect to the above composition of geometric conjugations*, for which we need to show the following:

Proposition 2.1.2. Given four geometric functors $D: \mathbf{D}_+ \to \mathbf{D}_-$, $E: \mathbf{E}_+ \to \mathbf{E}_-$, $F: \mathbf{F}_+ \to \mathbf{F}_-$, and $G: \mathbf{G}_+ \to \mathbf{G}_-$, let $(H_-, H_+, \alpha): D \to E$, $(K_-, K_+, \beta): E \to F$, and $(L_-, L_+, \gamma): F \to G$ be geometric conjugations*. Then

$$
\begin{aligned}
&((L_-, L_+, \gamma) \circ (K_-, K_+, \beta)) \circ (H_-, H_+, \alpha) \\
&= (L_-, L_+, \gamma) \circ ((K_-, K_+, \beta) \circ (H_-, H_+, \alpha))
\end{aligned}
$$

Proof. We note that

$$((L_-, L_+, \gamma) \circ (K_-, K_+, \beta)) \circ (H_-, H_+, \alpha)$$
$$= (L_- \circ K_-, L_+ \circ K_+, (L_- \circ \beta) \cdot (\gamma \circ K_+)) \circ (H_-, H_+, \alpha)$$
$$= (L_- \circ K_- \circ H_-, L_+ \circ K_+ \circ H_+, (L_- \circ K_- \circ \alpha) \cdot (((L_- \circ \beta) \cdot (\gamma \circ K_+)) \circ H_+))$$
$$= (L_- \circ K_- \circ H_-, L_+ \circ K_+ \circ H_+, (L_- \circ K_- \circ \alpha) \cdot (L_- \circ \beta \circ H_+) \cdot (\gamma \circ K_+ \circ H_+))$$
[by (1.1.8)]

while we have that

$$(L_-, L_+, \gamma) \circ ((K_-, K_+, \beta) \circ (H_-, H_+, \alpha))$$
$$= (L_-, L_+, \gamma) \circ (K_- \circ H_-, K_+ \circ H_+, (K_- \circ \alpha) \cdot (\beta \circ H_+))$$
$$= (L_- \circ K_- \circ H_-, L_+ \circ K_+ \circ H_+, (L_- \circ ((K_- \circ \alpha) \cdot (\beta \circ H_+))) \cdot (\gamma \circ K_+ \circ H_+))$$
$$= (L_- \circ K_- \circ H_-, L_+ \circ K_+ \circ H_+, (L_- \circ K_- \circ \alpha) \cdot (L_- \circ \beta \circ H_+) \cdot (\gamma \circ K_+ \circ H_+))$$
[by (1.1.7)]

Therefore the desired equality holds. ■

Corollary 2.1.3. $\mathbf{Geom}_2^*$ is a category.

Given three geometric functors E: $\mathbf{E}_+ \to \mathbf{E}_-$, F: $\mathbf{F}_+ \to \mathbf{F}_-$, and G: $\mathbf{G}_+ \to \mathbf{G}_-$, two geometric conjugations* (H_-, H_+, α) and (K_-, K_+, β) from E to F, and two geometric conjugations* (H'_-, H'_+, α') and (K'_-, K'_+, β') from F to G, we define the horizontal composite of two geometric transformations* (σ_-, σ_+): $(H_-, H_+, \alpha) \to (K_-, K_+, \beta)$ and (τ_-, τ_+): $(H'_-, H'_+, \alpha') \to (K'_-, K'_+, \beta')$, in notation $(\tau_-, \tau_+) \circ (\sigma_-, \sigma_+)$, to be $(\tau_- \circ \sigma_-, \tau_+ \circ \sigma_+)$, for which we have to show the following.

Proposition 2.1.4. $(\tau_-, \tau_+) \circ (\sigma_-, \sigma_+)$ is a geometric transformation* from the geometric conjugation* $(H'_-, H'_+, \alpha') \circ (H_-, H_+, \alpha)$: $E \to G$ to the geometric conjugation* $(K'_-, K'_+, \beta') \circ (K_-, K_+, \beta)$: $E \to G$.

Proof. Since

$$(H'_-, H'_+, \sigma') \circ (H_-, H_+, \alpha) = (H'_- \circ H_-, H'_+ \circ H_+, (H'_- \circ \alpha) \cdot (\alpha' \circ H_+))$$
$$(K'_-, K'_+, \beta') \circ (K_-, K_+, \beta) = (K'_- \circ K_-, K'_+ \circ K_+, (K'_- \circ \beta) \cdot (\beta' \circ K_+))$$

we have to show that

$$((K'_- \circ \beta) \cdot (\beta' \circ K_+)) \cdot (G \circ \tau_+ \circ \sigma_+) = (\tau_- \circ \sigma_- \circ E) \cdot (H'_- \circ \alpha) \cdot (\alpha' \circ H_+)$$

which follows from the following calculation:

$$(\tau_- \circ \sigma_- \circ E)\cdot(H'_- \circ \alpha)\cdot(\alpha' \circ H_+)$$

$$= (\tau_- \circ \sigma_- \circ E)\cdot(\mathrm{id}_{H'_-} \circ \alpha)\cdot(\alpha' \circ H_+) \qquad [(1.3.4)]$$

$$= (\tau_- \circ ((\sigma_- \circ E)\cdot(\alpha))\cdot(\alpha' \circ H_+) \qquad [(1.3.5)]$$

$$= (\tau_- \circ (\beta\cdot(F \circ \sigma_+)))\cdot(\alpha' \circ H_+) \qquad [(2.1.3)]$$

$$= (K'_- \circ (\beta\cdot(F \circ \sigma_+)))\cdot(\tau_- \circ F \circ H_+)\cdot(\alpha' \circ H_+) \qquad [\text{Definition of } \circ]$$

$$= (K'_- \circ (\beta\cdot(F \circ \sigma_+)))\cdot(((\tau_- \circ F)\cdot\alpha') \circ H_+) \qquad [(1.1.8)]$$

$$= (K'_- \circ (\beta\cdot(F \circ \sigma_+)))\cdot((\beta'\cdot(G \circ \tau_+)) \circ H_+) \qquad [(2.1.3)]$$

$$= (K'_- \circ \beta)\cdot(K'_- \circ (F \circ \sigma_+))\cdot(\beta' \circ H_+)\cdot(G \circ \tau_+ \circ H_+) \qquad [(1.1.7), (1.1.8)]$$

$$= (K'_- \circ \beta)\cdot(\beta' \circ K_+)\cdot(G \circ K'_+ \circ \sigma_+)\cdot(G \circ \tau_+ \circ H_+) \qquad [(1.3.1)]$$

$$= (K'_- \circ \beta)\cdot(\beta' \circ K_+)\cdot(G \circ ((K'_+ \circ \sigma_+)\cdot(\tau_+ \circ H_+))) \qquad [(1.1.7)]$$

$$= (K'_- \circ \beta)\cdot(\beta' \circ K_+)\cdot(G \circ \tau_+ \circ \sigma_+) \qquad [\text{Definition of } \circ]$$

We recapitulate the discussion so far.

Theorem 2.1.5. $\mathbf{GEOM}_2^*$ is a 2-category with respect to vertical and horizontal compositions of geometric transformations*.

Now we discuss a variant of the 2-category $\mathbf{GEOM}_2^*$. Given two geometric functors E: $\mathbf{E}_+ \to \mathbf{E}_-$ and F: $\mathbf{F}_+ \to \mathbf{F}_-$, a *geometric conjugation*$_*$ from E to F is a triple (H_-, H_+, α) of two geometric functors $H_\pm$: $\mathbf{E}_\pm \to \mathbf{F}_\pm$ and a natural transformation α: $H_- \circ E \to F \circ H_+$. Given two geometric conjugations$_*$ (H_-, H_+, α) and (K_-, K_+, β) from E to F, a *geometric transformation*$_*$ from (H_-, H_+, α) to (K_-, K_+, β) is a pair (σ_-, σ_+) of natural transformations $\sigma_\pm$: $H_\pm \to K_\pm$ subject to the following condition:

(2.1.7) $\beta\cdot(\sigma_- \circ E) = (F \circ \sigma_+)\cdot\alpha.$

A due variant of the discussion leading to Theorem 2.1.5 establishes the following:

Theorem 2.1.6. The set $\mathbf{GEOM}_*^2$ of geometric functors of small$_2$ toposes (as objects), geometric conjugations$_*$ (as morphisms), and geometric transformations$_*$ (as 2-arrows) is a 2-category with respect to the following operations:

(2.1.8) Given three geometric functors E: $\mathbf{E}_+ \to \mathbf{E}_-$, F: $\mathbf{F}_+ \to \mathbf{F}_-$, and G: $\mathbf{G}_+ \to \mathbf{G}_-$, the composite of geometric conjugations$_*$ (H_-, H_+, α): $E \to$

F and (K_-, K_+, β): $F \to G$, in notation $(K_-, K_+, \beta) \circ (H_-, H_+, \alpha)$, is defined to be the geometric conjugation$_*$ $(K_- \circ H_-, K_+ \circ H_+, (\beta \circ H_+) \cdot (K_- \circ \alpha))$.

(2.1.9) Given three geometric conjugations$_*$ (H_-, H_+, α), (K_-, K_+, β), and (L_-, L_+, γ) from a geometric functor E of small$_2$ toposes to a geometric functor F of small$_2$ toposes, we define the vertical composite of geometric transformations$_*$ (σ_-, σ_+): $(H_-, H_+, \alpha) \to (K_-, K_+, \beta)$ and (τ_-, τ_+): $(K_-, K_+, \beta) \to (L_-, L_+, \gamma)$, denoted by $(\tau_-, \tau_+) \cdot (\sigma_-, \sigma_+)$, to be $(\tau_- \cdot \sigma_-, \tau_+ \cdot \sigma_+)$, which is to be put down as a geometric transformation$_*$ from (H_-, H_+, α) to (K_-, K_+, γ) by the same token as in Proposition 2.1.1.

(2.1.10) Given three geometric functors E: $\mathbf{E}_+ \to \mathbf{E}_-$, F: $\mathbf{F}_+ \to \mathbf{F}_-$, and G: $\mathbf{G}_+ \to \mathbf{G}_-$, two geometric conjugations$_*$ (H_-, H_+, α) and (K_-, K_+, β) from E to F, and two geometric conjugations$_*$ (H'_-, H'_+, α') and (K'_-, K'_+, β') from F to G, we define the horizontal composite of two geometric transformations$_*$ (σ_-, σ_+): $(H_-, H_+, \alpha) \to (K_-, K_+, \beta)$ and (τ_-, τ_+): $(H'_-, H'_+, \alpha') \to (K'_-, K'_+, \beta')$, in notation $(\tau_-, \tau_+) \circ (\sigma_-, \sigma_+)$, to be $(\tau_- \circ \sigma_-, \tau_+ \circ \sigma_+)$, which is to be put down as a geometric transformation from $(H'_-, H'_+, \alpha') \circ (H_-, H_+, \alpha)$ to $(K'_-, K'_+, \beta') \circ (K_-, K_+, \beta)$ by the same token as in Proposition 2.1.4.

We denote by $\mathbf{Geom}^2_*$ the category of geometric functors of small$_2$ toposes and geometric conjugations$_*$.

We close this subsection with a proposition connecting the two categories $\mathbf{Geom}^*_2$ and $\mathbf{Geom}^2_*$, for which we first need to fix some notation and terminology. Let us consider the following diagram in $\mathbf{Top}_2$, which is not assumed to be commutative at all:

$$\begin{array}{ccc}
\mathbf{E}_+ & \xrightarrow{E} & \mathbf{E}_- \\
{\scriptstyle H_+}\downarrow & & \downarrow{\scriptstyle H_-} \\
\mathbf{F}_+ & \xrightarrow{F} & \mathbf{F}_- \\
{\scriptstyle K_+}\downarrow & & \downarrow{\scriptstyle K_-} \\
\mathbf{G}_+ & \xrightarrow{G} & \mathbf{G}_-
\end{array}$$

Given natural transformations α: $F \circ H_+ \to H_- \circ E$ and β: $G \circ K_+ \to K_- \circ F$, we denote the third component of the composite $(K_-, K_+, \beta) \circ (H_-, H_+, \alpha)$ of geometric conjugations* within the category $\mathbf{Geom}^*_2$ by $\beta \circ^* \alpha$, which is surely a natural transformation from $G \circ K_+ \circ H_+$ to $K_- \circ H_- \circ E$.

Let us consider the following diagram in $\mathbf{Top}_2$, which is not assumed to be commutative at all:

$$\begin{array}{ccccc} \mathbf{E}_+ & \xrightarrow{H_+} & \mathbf{F}_+ & \xrightarrow{K_+} & \mathbf{G}_+ \\ {\scriptstyle E}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle G}\downarrow \\ \mathbf{E}_- & \xrightarrow[H_-]{} & \mathbf{F}_- & \xrightarrow[K_-]{} & \mathbf{G}_- \end{array}$$

Given natural transformations $\alpha: H_- \circ E \to F \circ H_+$ and $\beta: K_- \circ F \to G \circ K_+$, we denote the third component of the composite $(K_-, K_+, \beta) \circ (H_-, H_+, \alpha)$ of geometric conjugations$_*$ within the category $\mathbf{Geom}_*^2$ by $\beta \circ {}_*\alpha$, which is surely a natural transformation from $K_- \circ H_- \circ E$ to $G \circ K_+ \circ H_+$.

Proposition 2.1.7. Consider the following diagram within the category $\mathbf{Top}_2$, which is not assumed to be commutative at all:

$$\begin{array}{ccccc} \mathbf{E}_1 & \xrightarrow{E} & \mathbf{E}_2 & \xrightarrow{F} & \mathbf{E}_3 \\ {\scriptstyle D}\downarrow & & {\scriptstyle D'}\downarrow & & {\scriptstyle D''}\downarrow \\ \mathbf{F}_1 & \xrightarrow{E'} & \mathbf{F}_2 & \xrightarrow{F'} & \mathbf{F}_3 \\ {\scriptstyle G}\downarrow & & {\scriptstyle G'}\downarrow & & {\scriptstyle G''}\downarrow \\ \mathbf{G}_1 & \xrightarrow{E''} & \mathbf{G}_2 & \xrightarrow{F''} & \mathbf{G}_3 \end{array}$$

Given four natural transformations $\alpha: E' \circ D \to D' \circ E$, $\alpha': E'' \circ G \to G' \circ E'$, $\beta: F' \circ D' \to D'' \circ F$, and $\beta': F'' \circ G' \to G'' \circ F'$, we have

$$(2.1.11) \quad (\beta' \circ {}^*\beta) \circ {}_*(\alpha' \circ {}^*\alpha) = (\beta' \circ {}_*\alpha') \circ {}^*(\beta \circ {}_*\alpha)$$

Proof. This follows from the following calculation:

$$\begin{aligned} &(\beta' \circ {}^*\beta) \circ {}_*(\alpha' \circ {}^*\alpha) \\ &= ((G'' \circ \beta)\cdot(\beta' \circ D')) \circ {}_*((G' \circ \alpha)\cdot(\alpha' \circ D)) \\ &= (((G'' \circ \beta)\cdot(\beta' \circ D')) \circ E)\cdot(F'' \circ ((G' \circ \alpha)\cdot(\alpha' \circ D))) \\ &= (G'' \circ \beta \circ E)\cdot(\beta' \circ D' \circ E)\cdot(F'' \circ G' \circ \alpha)\cdot(F'' \circ \alpha' \circ D) \\ &\qquad [(1.1.7), (1.1.8)] \\ &= (G'' \circ \beta \circ E)\cdot(G'' \circ F' \circ \alpha)\cdot(\beta' \circ E' \circ D)\cdot(F'' \circ \alpha' \circ D) \\ &\qquad [(1.3.1)] \end{aligned}$$

$$= (G'' \circ ((\beta \circ E)\cdot(F' \circ \alpha)))\cdot(((\beta' \circ E')\cdot(F'' \circ \alpha')) \circ D)$$

[(1.1.7), (1.1.8)]

$$= ((\beta \circ E)\cdot(F' \circ \alpha)) \circ {*}((\beta' \circ E')\cdot(F'' \circ \alpha'))$$

$$= (\beta' \circ {}_*\alpha') \circ {*}(\beta \circ {}_*\alpha) \quad \blacksquare$$

2.2. Topology

Let us recall that a *topology* on a topos E with a subobject classifier t: $1 \to \Omega$ is a morphism j: $\Omega \to \Omega$ abiding by the following identities:

(2.2.1) $j \circ t = t$.
(2.2.2) $j \circ j = j$.
(2.2.3) $j \circ \wedge = \wedge \circ (j \times j)$.

A *universal closure operation* on **E** is an assignment to each subobject $X \rightarrowtail a$ of another subobject $\bar{x} \rightarrowtail a$ (called the *closure* of x in a) abiding by the following conditions:

(2.2.4) $x \subseteq \bar{x}$.
(2.2.5) If $x \subseteq y$, then $\bar{x} \subseteq \bar{y}$.
(2.2.6) $\bar{\bar{x}} = \bar{x}$.
(2.2.7) $f^{-1}(\bar{x}) = \overline{f^{-1}(x)}$ for any morphism f: $b \to a$.

It is well known that there is a bijection between the topologies on **E** and the universal closure operations on **E** (Borceux, 1994, Vol. 3, Proposition 9.1.3).

A topos endowed with a topology is called a *localized topos*. Let (**E**, j) be a localized topos. A subobject $x \rightarrowtail a$ is said to be *dense* if $\bar{x} = a$. An object b of **E** is called a *j-sheaf* if for any dense subobject s: $x \rightarrowtail a$ and any morphism f: $x \to b$ there exists a unique morphism g: $a \to b$ such that $f = g \circ s$. The full subcategory of E whose objects are all j-sheaves is denoted by **Sh**(E,;j), for which the following associated sheaf functor theorem is fundamental.

Theorem 2.2.1. The inclusion functor $\mathbf{i}_j$: **Sh**(**E**, j) $\to$ **E** has a left adjoint $\mathbf{a}_j$: **E** $\to$ **Sh**(**E**, j) which is left exact. The category **Sh**(**E**, j) is a topos. Therefore the pair ($\mathbf{i}_j$, a_j) forms a geometric morphism from **Sh**(E, j) to **E**.

For a proof of the above theorem, the reader is referred to Borceux (1994, Vol. 3, Theorems 9.2.10, 9.2.11, and 9.3.8).

A geometric functor E: $\mathbf{E}_+ \to \mathbf{E}_-$ is said to be *localized* if both of the toposes $\mathbf{E}_\pm$ are localized with topologies $j_\pm$ and E satisfies the following condition:

(2.2.8) $E(\bar{x}) \subseteq \overline{E(x)}$ for any subobject $x \rightarrowtail a$ in $\mathbf{E}_+$.

In this case we say that E: $(\mathbf{E}_+, j_+) \to (\mathbf{E}_-, j_-)$ is a localized geometric functor or that E is a localized geometric functor from the localized topos $(\mathbf{E}_+, j_+)$ to the localized topos $(\mathbf{E}_-, j_-)$.

Theorem 2.2.2. Let E: $(\mathbf{E}_+, j_+) \to (\mathbf{E}_-, j_-)$ be a localized geometric functor. Then the functor

$$\mathbf{a}_{j_-} \circ E \circ \mathbf{i}_{j_+}: \mathbf{Sh}(\mathbf{E}_+, j_+) \to \mathbf{Sh}(\mathbf{E}_-, j_-)$$

is a geometric functor to be denoted by $\mathbf{Sh}^*(E, j_-, j_+)$.

For the proof of the above theorem the reader is referred to Nishimura (1996a, Theorem 2.1.5).

Theorem 2.2.3. Let E: $(\mathbf{E}_1, j_1) \to (\mathbf{E}_2, j_2)$ and F: $(\mathbf{E}_2, j_2) \to (\mathbf{E}_3, j_3)$ be localized geometric functors. Then $G = F \circ E$ is a localized geometric functor from the localized topos (E_1, j_1) to the localized topos (E_3, j_3), and the functors $\mathbf{Sh}^*(G, j_3, j_1)$ and $\mathbf{Sh}^*(F, j_3, j_2) \circ \mathbf{Sh}^*(E, j_2, j_1)$ are naturally isomorphic.

For the proof of the above theorem the reader is referred to Nishimura (1996a, Theorem 2.1.6). As in Nishimura (n.d.-a, Theorem 2.1.7), Theorem 2.2.3 implies the following:

Theorem 2.2.4. Let E: $(\mathbf{E}_+, j_+) \to (\mathbf{E}_-, j_-)$ be a localized geometric functor. Then the functors $\mathbf{a}_{j_-} \circ E$ and $\mathbf{Sh}^*(E, j_-, j_+) \circ \mathbf{a}_{j_+}$ are naturally isomorphic.

2.3. Grothendieck Topology

Recall that a *Grothendieck topology* on a category $\mathbf{C}$ is an assignment L to each $a \in \mathrm{Ob}\,\mathbf{C}$ of a family $\mathbf{L}(a)$ of subfunctors of $\mathbf{C}(?, a)$ obeying the following conditions:

(2.3.1) $\mathbf{C}(?, a) \in \mathbf{L}(a)$.

(2.3.2) Let f: $b \to a$ be a morphism of $\mathbf{C}$. Let R and R_f be subfunctors of $\mathbf{C}(?, a)$ and $\mathbf{C}(?, b)$ respectively. If the following square

$$\begin{array}{ccc} R_f & \longrightarrow & R \\ \downarrow & & \downarrow \\ \mathbf{C}(?, b) & \xrightarrow[\mathbf{C}(?, f)]{} & \mathbf{C}(?, a) \end{array}$$

is a pullback diagram and $\mathbf{R} \in \mathbf{L}(a)$, then $R_f \in \mathbf{L}(b)$.

(2.3.3) Let $a \in \text{Ob}\ \mathbf{C}$, R a subfunctor of $\mathbf{C}(?, a)$, and $S \in \mathbf{L}(a)$. If for any $b \in \text{Ob}\ \mathbf{C}$ and any $f: b \to a \in S(b)$ we have $R_f \in \mathbf{L}(b)$ with R_f being defined as in (2.3.2), then we have $R \in \mathbf{L}(a)$.

A subfunctor R of $\mathbf{C}(?, a)$ for some $a \in \text{Ob}\ \mathbf{C}$ is usually identified with a *sieve* on a, which is by definition a set of morphisms f of $\mathbf{C}$ with codomain a which is closed under right composition. Given a Grothendieck topology $\mathbf{L}$ on $\mathbf{C}$ and $a \in \text{Ob}\ \mathbf{C}$, a set S of morphisms of $\mathbf{C}$ with codomain a is said to $\mathbf{L}$*-cover* a if $\text{Siv}(S) = \{g: b \to a \in \text{Mor}\ \mathbf{C} \mid g = f \circ h$ for some $f \in S$ and some $h \in \text{Mor}\ \mathbf{C}\} \in \mathbf{L}(a)$.

The following well-known theorem signifies that the notion of a topology on a topos is a good generalization of a Grothendieck topology on a category.

Theorem 2.3.1. Let $\mathbf{C}$ be a small$_i$ category ($i = 0, 1, 2$). Then there is a bijection between the topologies on the topos $\mathbf{PreSh}_i(\mathbf{C})$ and the Grothendieck topologies on the category $\mathbf{C}$.

For a proof of the above theorem the reader is referred to Borceux (1994, Vol. 3, Proposition 9.1.2).

A pair $(\mathbf{C}, \mathbf{L})$ of a small$_i$ category $\mathbf{C}$ and a Grothendieck topology $\mathbf{L}$ on $\mathbf{C}$ is called a *(small$_i$) site*. A site $(\mathbf{C}, \mathbf{L})$ is said to be *finitely complete* if the category $\mathbf{C}$ is finitely complete. For a small$_i$ site $(\mathbf{C}, \mathbf{L})$, the topology on $\mathbf{PreSh}_i(\mathbf{C})$ corresponding to the Grothendieck topology $\mathbf{L}$ under Theorem 2.3.1 is denoted by $j[\mathbf{L}]$, and the topos $\mathbf{Sh}(\mathbf{PreSh}_i(\mathbf{C}), j[\mathbf{L}])$ is denoted by $\mathbf{Sh}(\mathbf{C}, \mathbf{L})$, and is called the Grothendieck topos associated with the site $(\mathbf{C}, \mathbf{L})$. The elements of $\mathbf{Sh}(\mathbf{C}, \mathbf{L})$ are called $\mathbf{L}$*-sheaves*. If $\mathbf{L}$ happens to be the largest Grothendieck topology with respect to which all the representable functors on $\mathbf{C}$ are $\mathbf{L}$-sheaves, then $\mathbf{L}$ is called the *canonical Grothendieck topology* on $\mathbf{C}$. Giraud's theorem (MacLane and Moerdijk, 1992, Appendix, §4, Corollary 2) implies that for any small$_i$ site $(\mathbf{C}, \mathbf{L})$ there exists a finitely complete small$_i$ site $(\mathbf{C}', \mathbf{L}')$ such that their Grothendieck toposes $\mathbf{Sh}_i(\mathbf{C}, \mathbf{L})$ and $\mathbf{Sh}_i(\mathbf{C}', \mathbf{L}')$ are equivalent.

Since every poset can naturally be regarded as a category (MacLane, 1971, p. 11), a complete Heyting algebra $\mathbf{H}$ in $\mathbf{Ens}_i$ can be put down as a category. We denote $\mathbf{Sh}_i(\mathbf{H}, \mathbf{L}[\mathbf{H}])$ simply by $\mathbf{Sh}_i(\mathbf{H})$, where $\mathbf{L}[\mathbf{H}]$ is the canonical Grothendieck topology on $\mathbf{H}$.

The following theorem is well known.

Theorem 2.3.2. For any small$_i$ functor $\varphi: \mathbf{C}_+ \to \mathbf{C}_-$, there exists, up to natural isomorphisms, a unique functor $\mathbf{PreSh}_i^*(\varphi)$: $\mathbf{PreSh}_i(\mathbf{C}_+) \to$ **Pre-**

$\mathbf{Sh}_i(\mathbf{C}_-)$ preserving small$_i$ colimits and making the following diagram commutative:

$$\begin{array}{ccc} \mathbf{PreSh}_i(\mathbf{C}_+) & \xrightarrow{\mathbf{PreSh}_i^*(\varphi)} & \mathbf{PreSh}_i(\mathbf{C}_-) \\ {\scriptstyle \mathbf{y}}\uparrow & & \uparrow{\scriptstyle \mathbf{y}} \\ \mathbf{C}_+ & \xrightarrow[\varphi]{} & \mathbf{C}_- \end{array}$$

For a proof of the above theorem the reader is referred to MacLane and Moerdijk (1992, Chapter I, §5, Corollary 4).

Theorem 2.3.3. In the above theorem, if $\mathbf{C}_+$ is finitely complete and φ preserves finite limits, then $\mathbf{PreSh}_i^*(\varphi)$ is a geometric functor.

Proof. This follows from Theorem 2 of MacLane and Moerdijk (1992, Chapter I, §5) and Theorem 17.1.6(e) of Schubert (1972). ∎

Given two finitely complete sites $(\mathbf{C}_\pm, \mathbf{L}_\pm)$, a *morphism of sites* from $(\mathbf{C}_+, \mathbf{L}_+)$ to $(\mathbf{C}_-, \mathbf{L}_-)$ is a functor φ from the category $\mathbf{C}_+$ to the category $\mathbf{C}_-$ preserving finite limits and covers, where φ is said to *preserve covers* provided that for any $a \in \mathrm{Ob}\,\mathbf{C}_+$ and any $S \in \mathbf{L}_+(a)$, $\varphi(S)$ $\mathbf{L}_-$-covers $\varphi(a)$.

Theorem 2.3.4. For any morphism φ: $(\mathbf{C}_+, \mathbf{L}_+) \to (\mathbf{C}_-, \mathbf{L}_-)$ of finitely complete small$_i$ sites, the geometric functor $\mathbf{PreSh}_i^*(\varphi)$: $\mathbf{PreSh}_i(\mathbf{C}_+) \to \mathbf{PreSh}_i(\mathbf{C}_-)$ is a localized geometric functor from $(\mathbf{PreSh}_i(\mathbf{C}_+), j[\mathbf{L}_+])$ to $(\mathbf{PreSh}_i(\mathbf{C}_-), j[\mathbf{L}_-])$, thereby inducing its associated geometric functor

$$\mathbf{Sh}_i^*(\mathbf{PreSh}_i^*(\varphi), j[\mathbf{L}_-], j[\mathbf{L}_+]): \mathbf{Sh}_i(\mathbf{C}_+, \mathbf{L}_+) \to \mathbf{Sh}_i(\mathbf{C}_-, \mathbf{L}_-)$$

to be denoted by $\mathbf{Sh}_i^*(\varphi, \mathbf{L}_-, \mathbf{L}_+)$.

Proof. This follows from Theorem 2.2.8 of Nishimura (n.d.-a) and Theorem 2.2.2. ∎

2.4. Classifying Toposes

We denote by $\mathbf{fp}\text{-}\mathbf{Rng}_0$ the category of finitely presented small$_0$ rings and homomorphisms. Recall that a ring is said to be *finitely presented* if it is describable by a finite generator together with a finite set of polynomial equations.

Each topos $\mathbf{E}$ enjoys intuitionistic logic, and is susceptible of first-order interpretations (MacLane and Moerdijk, 1992, Chapter X, §2). In particular, we can consider the category of rings and homomorphisms within the topos $\mathbf{E}$ to be denoted by $\mathbf{Rng}(\mathbf{E})$.

Theorem 2.4.1. For any small_1-cocomplete small_2 topos $\mathbf{E}$, there is an equivalence of categories

$$\mathbf{TOP}_2(\mathbf{PreSh}_1((\mathbf{fp\text{-}Rng}_0)^{op}), \mathbf{E}) \xrightarrow{\sim} \mathbf{Rng}(\mathbf{E})$$

which is natural in $\mathbf{E}$.

For a proof of the above theorem the reader is referred to MacLane and Moerdijk (1992, Chapter VIII, §5, Theorem 2).

Several equivalent definitions of a local ring in classical mathematics have their nonequivalent ramifications in intuitionistic mathematics. For a standard geometric definition of a local ring in a topos $\mathbf{E}$, the reader is referred to MacLane and Moerdijk (1992, Chapter VIII, §6). Given a topos $\mathbf{E}$, we denote by $\mathbf{LocRng}(\mathbf{E})$ the category of local rings and homomorphisms in $\mathbf{E}$. We denote by $\mathbf{L}_{\mathrm{Zar}}$ the minimal Grothendieck topology on $(\mathbf{fp\text{-}Rng}_0)^{op}$ subject to the following condition:

(2.4.1) For any finitely presented small_0 ring A and any elements $a_1, \ldots, a_n \in A$ such that $a_1 + \cdots + a_n = 1$, the dual family of canonical projections $A \to A[a_i^{-1}]$ $(1 \leq i \leq n)$ L_{Zar}-covers A.

Theorem 2.4.2. For any small_1-cocomplete small_2 topos $\mathbf{E}$, there is an equivalence of categories

$$\mathbf{TOP}_2(\mathbf{Sh}_1((\mathbf{fp\text{-}Rng}_0)^{op}, \mathbf{L}_{\mathrm{Zar}}), \mathbf{E}) \xrightarrow{\sim} \mathbf{LocRng}(\mathbf{E})$$

which is natural in $\mathbf{E}$.

For a proof of the above theorem, the reader is referred to MacLane and Moerdijk (1992, Chapter VIII, §6, Theorem 3).

2.5. Boolean Localic Toposes

A topos $\mathbf{E}$ is called *localic*$_i$ $(i = 0, 1, 2)$ if it obeys the following conditions:

(2.5.1) There exists a geometric morphism from $\mathbf{E}$ to $\mathbf{Ens}_i$.
(2.5.2) The subobjects of a terminal object 1 of $\mathbf{E}$ form a class of generators.

Any complete Heyting algebra $\mathbf{H}$ in the topos $\mathbf{Ens}_i$ gives rise to a localic_i topos $\mathbf{Sh}_i(\mathbf{H})$, and the localic_i toposes are characterized by the following theorem.

Theorem 2.5.1. For any localic_i topos $\mathbf{E}$, there exists an essentially unique complete Heyting algebra $\mathbf{H}$ in $\mathbf{Ens}_i$ such that $\mathbf{Sh}_i(\mathbf{H})$ is equivalent to $\mathbf{E}$.

For a proof of the above theorem, the reader is referred, e.g., to Johnstone (1977, Theorem 5.37) and Bell (1988, Theorem 6.4 and Corollary 6.5).

A topos **E** is said to be *Boolean* if its internal logic is classical. Any complete Boolean algebra **B** in the topos $\mathbf{Ens}_i$ gives rise to a Boolean localic$_i$ topos $\mathbf{Sh}_i(\mathbf{H})$, and the following theorem is an easy consequence of Theorem 2.5.1.

Theorem 2.5.2. For any Boolean localic$_i$ topos **E**, there exists an essentially unique complete Boolean algebra **B** in $\mathbf{Ens}_i$ such that $\mathbf{Sh}_i(\mathbf{B})$ is equivalent to **E**.

There are other representations of Boolean localic$_i$ toposes. One is Scott and Solovay's construction of Boolean-valued models $V_i^{(\mathbf{B})}$ in axiomatic set theory. Each element of $V_i^{(\mathbf{B})}$ is a function ξ whose domain $\mathscr{D}(\xi)$ is a subset of $V_i^{(\mathbf{B})}$ and which takes values in **B**. The elements of $V_i^{(\mathbf{B})}$ are constructed by transfinite induction on ordinals in V_i. For the details of the construction of $V_i^{(\mathbf{B})}$ the reader is referred to any standard textbook on axiomatic set theory such as Jech (1978) or Takeuti and Zaring (1973). What is important to notice for our later developments is only that $V_i^{(\mathbf{B})}$ is a model of ZFC and that the category of sets and functions within $V_i^{(\mathbf{B})}$ is equivalent to $\mathbf{Sh}_i(\mathbf{B})$.

Other highly useful representations of a Boolean localic$_i$ topos is the category $\underline{\mathbf{BEns}}_i(\mathbf{B})$ of small$_i$ **B**-valued sets and the category $\mathbf{BEns}_i(\mathbf{B})$ of complete small$_i$ **B**-valued sets for some complete Boolean algebra **B** in $\mathbf{Ens}_i$. The latter category is a full subcategory of the former, and the inclusion functor of the latter category into the former is an equivalence of categories. Given a Boolean locale X, we write $\underline{\mathbf{BEns}}_i(\mathrm{X})$ and $\mathbf{BEns}_i(\mathrm{X})$ for $\underline{\mathbf{BEns}}_i(\mathscr{P}(\mathrm{X}))$ and $\mathbf{BEns}_i(\mathscr{P}(\mathrm{X}))$, respectively. An object of $\mathbf{BEns}_i(\mathrm{X})$ is called a (*small*$_i$) X-set, and a morphism from an X-set $(\mathscr{U}, [\![\cdot = \cdot]\!]^{\mathscr{U}})$ to an X-set $(\mathscr{V}, [\![\cdot = \cdot]\!]^{\mathscr{V}})$ can be represented by an X-*function* from the former to the latter, which is a function $\mathscr{f}$ from $\mathscr{U}$ to $\mathscr{V}$ subject to the following conditions:

(2.5.3) $[\![x = y]\!]^{\mathscr{U}} \leq [\![\mathscr{f}(x) = \mathscr{f}(y)]\!]^{\mathscr{V}}$ for all $x, y \in \mathscr{U}$.
(2.5.4) $[\![\mathscr{f}(x) = \mathscr{f}(x)]\!]^{\mathscr{V}} \leq [\![x = x]\!]^{\mathscr{U}}$ for all $x \in \mathscr{U}$.

For a full treatment of the categories $\underline{\mathbf{BEns}}_i(\mathrm{X})$ and $\mathbf{BEns}_i(\mathrm{X})$ the reader is referred to Goldblatt (1979, §§11.9, 14.7).

Theorem 2.5.3. Let $\mathbf{E}_\pm$ be Boolean localic$_i$ toposes, so that they can be assumed to be of the form $\mathbf{Sh}_i(\mathbf{B}_\pm)$ for complete Boolean algebras $\mathbf{B}_\pm$ in $\mathbf{Ens}_i$. Then there is a bijective correspondence between the complete Boolean homomorphisms from $\mathbf{B}_+$ to $\mathbf{B}_-$ and the geometric functors from $\mathbf{E}_+$ to $\mathbf{E}_-$, where two geometric functors from $\mathbf{E}_+$ to $\mathbf{E}_-$ are identified so long as they are naturally isomorphic.

For a proof of the theorem, the reader is referred to MacLane and Moerdijk (1992, Chapter IX, §5, Proposition 2).

The theorem implies that a morphism $\mathfrak{f}: X_- \to X_+$ of Boolean locales induces a geometric functor $\mathfrak{f}^*_{\mathbf{BEns}_i}: \mathbf{BEns}_i(X_+) \to \mathbf{BEns}_i(X_-)$.

Given a morphism $\mathfrak{f}: X_- \to X_+$ of Boolean locales and small_i $X_\pm$-sets $(\mathcal{U}_\pm, [\![\cdot = \cdot]\!]_{X_\pm})$, an $\mathfrak{f}$-*function* is a function $\mathscr{f}$ from $\mathcal{U}_+$ to $\mathcal{U}_-$ subject to the following conditions:

(2.5.5) $[\![x = y]\!]_{X_+} \leq [\![\mathscr{f}(x) = \mathscr{f}(y)]\!]_{X_-}$ for all $x, y \in \mathcal{U}_+$.
(2.5.6) $[\![\mathscr{f}(x) = \mathscr{f}(x)]\!]_{X_-} \leq [\![x = x]\!]_{X_+}$ for all $x \in \mathcal{U}$.

As remarked in Nishimura (1996a, Theorem 2.5.1), we have the following result.

Proposition 2.5.4. Given a morphism $\mathfrak{f}: X_- \to X_+$ of Boolean locales and small_i $X_\pm$-sets $(\mathcal{U}_\pm, [\![\cdot = \cdot]\!]_{X_\pm})$, there is a bijection between the $\mathfrak{f}$-functions from $(\mathcal{U}_+, [\![\cdot = \cdot]\!]_{X_+})$ to $(\mathcal{U}_-, [\![\cdot = \cdot]\!]_{X_-})$ and the X_--functions from $\mathfrak{f}^*_{\mathbf{BEns}_i}((\mathcal{U}_+, [\![\cdot = \cdot]\!]_{X_+}))$ to $(\mathcal{U}_-, [\![\cdot = \cdot]\!]_{X_-})$.

3. BOOLEANIZATION MACHINERY

3.1. Two Fundamental Transfer Principles

Topos theory is a natural generalization of set theory, though the underlying logic of a topos is not necessarily classical, but only intuitionistic. It is well known that Boolean localic_i toposes enjoy classical set theory in principle ($i = 0, 1, 2$). For those well conversant with logic, the easiest way to see this is to represent a given Boolean localic_i topos in the form of $V_i^{(\mathbf{B})}$ with a complete Boolean algebra $\mathbf{B}$ in V_i and then to proceed via the following fundamental theorem of Boolean-valued set theory.

Theorem 3.1.1. If θ is a theorem of ZFC, then so is $[\![\theta]\!]_{\mathbf{B}} = 1$ with respect to $V_i^{(\mathbf{B})}$.

For the proof of the above theorem, the reader is referred to any standard textbook on axiomatic set theory such as Jech (1978, Theorem 43) or Takeuti and Zaring (1973, Theorem 13.12). Since all classical mathematics (= mathematics based on classical logic), ranging from algebraic geometry to functional analysis, is considered to be developed within classical set theory, every concept and each theorem of classical mathematics can be transferred to $V_i^{(\mathbf{B})}$, which we would like to call the set-theoretic version of the *first Boolean transfer principle*. Indeed the principle lies at the hub of Boolean mathematics institutionalized by Takeuti (1978).

Our topos-theoretic version of first Boolean transfer principle should be weaker than its set-theoretic version, since toposes do not admit of interpretations of the full language of ZFC, but are susceptible only of Mitchell–Bénabou languages. In any case, any Boolean localic$_i$ topos is equivalent to the category of sets and functions in $V_i^{(\mathbf{B})}$. Therefore a prudent formulation of the topos-theoretic version of the first Boolean transfer principle goes as follows:

Theorem 3.1.2. Any valid statement of classical mathematics which is interpretable in toposes is valid in any Boolean localic$_i$ topos.

For a full treatment of topos theory by using a Mitchell–Bénabou language, the reader is referred to Bell (1988). For a readable account of the relationship between the set-theoretic and topos-theoretic foundations of mathematics, the reader is referred to MacLane and Moerdijk (1992, Chapter VI, §10). Since toposes are rich domains in which most of the meaningful statements of mathematics are interpretable, Theorem 3.1.2 implies that Boolean localic$_i$ toposes enjoy classical mathematics.

Let φ: $\mathbf{B}_+ \to \mathbf{B}_-$ be a complete Boolean homomorphism of complete Boolean algebras $\mathbf{B}_\pm$ of V_i. A function $\tilde{\varphi}$: $V_i^{(\mathbf{B}+)} \to V_i^{(\mathbf{B}-)}$ is defined by transfinite induction as follows:

$$(3.1.1)\ \tilde{\varphi}(\xi) = \{(\tilde{\varphi}(\eta), \varphi(\xi(\eta))) \mid \eta \in \mathscr{D}(\xi)\}$$

Now our second Boolean transfer principle is based on the following simple theorem of Boolean-valued set theory.

Theorem 3.1.3. If $\theta(x_1, \ldots, x_n)$ is a formula of the language of ZFC with free variable among $x_1, \ldots, x_n$ in which every quantifier is bounded, and if $\xi_1, \ldots, \xi_n$ are elements of $V_i^{(\mathbf{B}+)}$, then we have the following:

$$(3.1.2)\ [\![\theta(\tilde{\varphi}(\xi_1), \ldots, \tilde{\varphi}(\xi_n))]\!]_{\mathbf{B}_-} = \varphi([\![\theta(\xi_1, \ldots, \xi_n)]\!]_{\mathbf{B}_+}).$$

The proof of the above theorem is essentially the same as that of Theorem 13.18 of Takeuti and Zaring (1973).

Since many-sorted first-order languages are interpretable in toposes (MacLane and Moerdijk, 1992, Chapter X, §2) and the function $\tilde{\varphi}$ naturally induces a geometric functor corresponding to φ in Theorem 2.5.3, Theorem 3.1.3 implies the following, the *second Boolean transfer principle.*

Theorem 3.1.4. Let E: $\mathbf{E}_+ \to \mathbf{E}_-$ be a geometric functor between Boolean localic$_i$ toposes. Let T be a theory in a many-sorted first-order language. Then E naturally induces a functor E_T from the category $\mathbf{Mod}(\mathbf{E}_+; T)$ of models of T in $\mathbf{E}_+$ to the category $\mathbf{Mod}(\mathbf{E}_-; T)$ of models of T in $\mathbf{E}_-$.

Another way to see the validity of Theorem 3.1.4 is to see that the geometric morphisms determined by E are open (Nishimura, 1993, Theorem 2.13; MacLane and Moerdijk, 1992, Chapter IX, §7, Proposition 2) and then to take Corollary 4 of MacLane and Moerdijk (1992, Chapter X, §3) into consideration. We note in passing that if T is a geometric theory, then the conclusion of Theorem 3.1.4 holds for any geometric functor E: $\mathbf{E}_+ \to \mathbf{E}_-$ of toposes, for which the reader is referred to Corollary 6 of MacLane and Moerdijk (1992, Chapter X, §3). Therefore a geometric functor E: $\mathbf{E}_+ \to \mathbf{E}_-$ of toposes naturally gives rise to functors $E_{\mathbf{Cat}}$: $\mathbf{Cat(E_+)} \to \mathbf{Cat(E_-)}$, $E_{\mathbf{Rng}}$: $\mathbf{Rng(E_+)} \to \mathbf{Rng(E_-)}$, and $E_{\mathbf{LocRng}}$: $\mathbf{LocRng(E_+)} \to \mathbf{LocRng(E_-)}$, where $\mathbf{Cat(E_\pm)}$, $\mathbf{Rng(E_\pm)}$, and $\mathbf{LocRng(E_\pm)}$ stand for the categories of categories, rings, and local rings within the toposes $\mathbf{E}_\pm$ respectively. If the toposes $\mathbf{E}_\pm$ are Boolean localic$_i$, then it induces a functor $E_{\mathbf{fp\text{-}Rng}}$: $\mathbf{fp\text{-}Rng(E_+)} \to \mathbf{fp\text{-}Rng(E_-)}$, where $\mathbf{fp\text{-}Rng(E_\pm)}$ stand for the categories of finitely presented rings within the toposes $\mathbf{E}_\pm$.

On account of Theorem 2.5.3, a morphism f: $\mathrm{X}_- \to \mathrm{X}_+$ of Boolean locales gives rise to a geometric functor $f^*_{\mathbf{BEns}_i}$: $\mathbf{BEns}_i(\mathrm{X}_+) \to \mathbf{BEns}_i(\mathrm{X}_-)$. We denote the functors

$$(f^*_{\mathbf{BEns}_i})_{\mathbf{Cat}}\colon \mathbf{Cat}_i(\mathrm{X}_+) \to \mathbf{Cat}_i(\mathrm{X}_-)$$

$$(f^*_{\mathbf{BEns}_i})_{\mathbf{Rng}}\colon \mathbf{Rng}_i(\mathrm{X}_+) \to \mathbf{Rng}_i(\mathrm{X}_-)$$

$$(f^*_{\mathbf{BEns}_i})_{\mathbf{LocRng}}\colon \mathbf{LocRng}_i(\mathrm{X}_+) \to \mathbf{LocRng}_i(\mathrm{X}_-)$$

$$(f^*_{\mathbf{BEns}_i})_{\mathbf{fp\text{-}Rng}}\colon \mathbf{fp\text{-}Rng}_i(\mathrm{X}_+) \to \mathbf{fp\text{-}Rng}_i(\mathrm{X}_-)$$

by $f^*_{\mathbf{Cat}_i}$, $f^*_{\mathbf{Rng}_i}$, $f^*_{\mathbf{LocRng}_i}$, and $f^*_{\mathbf{fp\text{-}Rng}_i}$, respectively, where $\mathbf{Cat}_i(\mathrm{X}_\pm)$, $\mathbf{Rng}_i(\mathrm{X}_\pm)$, $\mathbf{LocRng}_i(\mathrm{X}_\pm)$, and $\mathbf{fp\text{-}Rng}_i(\mathrm{X}_\pm)$ stand for the categories $\mathbf{Cat(BEns}_i(\mathrm{X}_\pm))$, $\mathbf{Rng(BEns}_i(\mathrm{X}_\pm))$, $\mathbf{LocRng(BEns}_i(\mathrm{X}_\pm))$, and $\mathbf{fp\text{-}Rng(BEns}_i(\mathrm{X}_\pm))$, respectively.

3.2. Booleanized Category Theory

Let X be a Boolean locale with $\mathbf{B} = \mathscr{P}(\mathrm{X})$, which shall be fixed throughout this subsection. By simply interpreting the notion of a category within the topos $\mathbf{BEns}_i(\mathrm{X})$ ($i = 0, 1, 2$), we get the notion of a (*small$_i$*) X-*category*, which is externally a 6-tuple (Ob $\mathscr{C}$, Mor $\mathscr{C}$, $d_{\mathscr{C}}$, $r_{\mathscr{C}}$, $\mathrm{id}_{\mathscr{C}}$, $\circ_{\mathscr{C}}$), where:

(3.2.1) Ob $\mathscr{C}$ and More $\mathscr{C}$ are small$_i$ **X**-sets.

(3.2.2) $d_{\mathscr{C}}$ and $r_{\mathscr{C}}$ are **X**-functions from Mor $\mathscr{C}$ to Ob $\mathscr{C}$.

(3.2.3) $\mathrm{id}_{\mathscr{C}}$ is an **X**-function from Ob $\mathscr{C}$ to Mor $\mathscr{C}$ such that $[\![x = y]\!]^{\mathrm{Ob}\,\mathscr{C}} = [\![\mathrm{id}_{\mathscr{C}}(x) = \mathrm{id}_{\mathscr{C}}(y)]\!]^{\mathrm{Mor}\,\mathscr{C}}$ for all $x, y \in \mathrm{Ob}\,\mathscr{C}$.

(3.2.4) $\circ_{\mathscr{C}}$ is an **X**-function from

$$\mathrm{Mor}\,\mathscr{C} \times_{\mathrm{Ob}\,\mathscr{C}} \mathrm{Mor}\,\mathscr{C} = \{(g, f) \in \mathrm{Mor}\,\mathscr{C} \times \mathrm{Mor}\,\mathscr{C} \mid d_{\mathscr{C}}(g) = r_{\mathscr{C}}(f)\}$$

to Mor $\mathscr{C}$, where we note that $\mathrm{Mor}\,\mathscr{C} \times_{\mathrm{Ob}\,\mathscr{C}} \mathrm{Mor}\,\mathscr{C}$ is an **X**-subset of $\mathrm{Mor}\,\mathscr{C} \times_{\mathrm{X}} \mathrm{Mor}\,\mathscr{C}$.

(3.2.5) If we regard Ob $\mathscr{C}$ and Mor $\mathscr{C}$ as mere sets, then the 6-tuple (Ob $\mathscr{C}$, Mor $\mathscr{C}$, $d_{\mathscr{C}}$, $r_{\mathscr{C}}$, $\mathrm{id}_{\mathscr{C}}$, $\circ_{\mathscr{C}}$) is a category in the usual sense.

Unless confusion may arise, the subscript $\mathscr{C}$ in $d_{\mathscr{C}}$, $r_{\mathscr{C}}$, $\mathrm{id}_{\mathscr{C}}$, and $\circ_{\mathscr{C}}$ is usually omitted. Given $p \in \mathbf{B}$, the full subcategory of the category $\mathscr{C}$ whose objects are all $x \in$ Ob $\mathscr{C}$ with $[\![x = x]\!] \leq p$ can naturally be rated as an $\mathbf{X}_p$-category and is denoted by $\mathscr{C}\upharpoonright p$. Given $p \in \mathbf{B}$, the full subcategory of the category $\mathscr{C}$ whose objects are all $x \in$ Ob $\mathscr{C}$ with $[\![x = x]\!] = p$ is called the *p-slice* of the **X**-category $\mathscr{C}$ and is denoted by $\mathscr{C}[p]$. The objects and morphisms of $\mathscr{C}[1_X]$ are called *total objects* and *total morphisms* of $\mathscr{C}$.

Such fundamental notions of category theory as that of a functor and that of a natural transformation are easily interpretable within the topos $\mathbf{BEns}_i$(X) to yield externally the notion of an X-functor and that of a natural X-transformation. By way of example, an X-functor from an X-category $\mathscr{C}$ to an X-category $\mathscr{D}$ is a functor from the category $\mathscr{C}$ to the category $\mathscr{D}$ satisfying the following condition:

(3.2.6) The assignment $f \in$ Mor $\mathscr{C} \mapsto \mathscr{F}(f) \in$ Mor$\mathscr{D}$ is an X-function.

Given an X-functor $\mathscr{F}$: $\mathscr{C} \rightarrow \mathscr{D}$ of X-categories and $p \in \mathbf{B}$, we write $\mathscr{F}[p]$ for the functor from the category $\mathscr{C}[p]$ to the category $\mathscr{D}[p]$ induced by $\mathscr{F}$.

Given two X-functors $\mathscr{F}$, $\mathscr{G}$: $\mathscr{C} \rightarrow \mathscr{D}$ of X-categories, a natural X-transformation is a natural transformation α from the functor $\mathscr{F}$ to the functor $\mathscr{G}$ subject to the following condition:

(3.2.7) The assignment $x \in$ Ob $\mathscr{C} \mapsto \alpha_x \in$ Mor$\mathscr{D}$ is an X-function.

Given two X-functors $\mathscr{F}$, $\mathscr{G}$: $\mathscr{C} \rightarrow \mathscr{D}$ of X-categories, a natural X-transformation α:$\mathscr{F} \rightarrow \mathscr{G}$, and $p \in \mathbf{B}$, we write $\alpha[p]$ for the natural transformation from the functor $\mathscr{F}[p]$ to the functor $\mathscr{G}[p]$ induced by α.

For a fuller treatment of such considerations, the reader is referred to Nishimura (1995c).

The interpretation of the notion of finiteness within the topos $\mathbf{BEns}_i$(X) gives rise to that of X-*finiteness* externally. A small$_i$ X-category $\mathscr{C}$ is called *small$_i$-X-complete* (*X-finitely X-complete*, resp.) if it is complete (finitely complete, resp.) internally within the topos $\mathbf{BEns}_i$(X). Now it is easy to see the following result:

Proposition 3.2.1. Let $\mathscr{C}$ be a small$_i$ X-category such that $\mathscr{C}[1_X]$ has a terminal object. Then the X-category $\mathscr{C}$ is small$_i$-X-complete (X-finitely X-complete, resp.) iff the category $\mathscr{C}[1_X]$ is small$_i$-complete (finitely complete, resp.).

A small_i X-category $\mathscr{C}$ is called *small_i-X-cocomplete* (*X-finitely X-cocomplete*, resp.) if it is cocomplete (finitely cocomplete, resp.) internally within the topos $\mathbf{BBns}_i(\mathrm{X})$. It is easy to see that

Proposition 3.2.2. Let $\mathscr{C}$ be a small_i X-category such that $\mathscr{C}[1_\mathrm{X}]$ has an initial object. Then the X-category $\mathscr{C}$ is small_i-X-cocomplete (X-finitely X-cocomplete, resp.) iff the category $\mathscr{C}[1_\mathrm{X}]$ is small_i-cocomplete (finitely cocomplete, resp.).

By interpreting of a limit (colimit, resp.) in the topos $\mathbf{BEns}_i(\mathrm{X})$, we get the notion of an *X-limit* (*X-colimit*, resp.), for which we have the following two propositions.

Proposition 3.2.3. Let $\mathscr{C}$ be a small_i X-category such that $\mathscr{C}[1_\mathrm{X}]$ has a terminal object. Let $\mathscr{F}$ be an X-functor from $\mathscr{C}$ to a small_i X-category $\mathscr{D}$. Then X-functor $\mathscr{F}$ preserves small_i X-limits (X-finite X-limits, resp.) iff the functor $\mathscr{F}[1_\mathrm{X}]$ preserves small_i limits (finite limits, resp.).

Proposition 3.2.4. Let $\mathscr{C}$ be a small_i X-category such that $\mathscr{C}[1_\mathrm{X}]$ has an initial object. Let $\mathscr{F}$ be an X-functor from $\mathscr{C}$ to a small_i X-category $\mathscr{D}$. Then X-functor $\mathscr{F}$ preserves small_i X-colimits (X-finite X-colimits, resp.) iff the functor $\mathscr{F}[1_\mathrm{X}]$ preserves small_i colimits (finite colimits, resp.).

By interpreting the notion of a left (right, resp.) adjoint of a functor within the topos $\mathbf{BEns}_i(\mathrm{X})$, we get the notion of a *left* (*right*, resp.) *X-adjoint* of an X-functor, for which it is easy to see the following.

Proposition 3.2.5. An X-functor $\mathscr{F}$: $\mathscr{C} \to \mathscr{D}$ of small_i X-categories with $\mathscr{C}[1_\mathrm{X}]$ being nonempty has a left (right, resp.) X-adjoint iff the functor $\mathscr{F}[1_\mathrm{X}]$: $\mathscr{C}[1_\mathrm{X}] \to \mathscr{D}[1_\mathrm{X}]$ has a left (right, resp.) adjoint.

3.3. Relations Between Two Booleanized Categories

Let f: $\mathrm{X}_- \to \mathrm{X}_+$ be a morphism of Boolean locales, which shall be fixed throughout this subsection.

Given $\mathrm{X}_\pm$-categories $\mathscr{C}_\pm$, a functor $\mathscr{F}$ from the category $\mathscr{C}_+$ to the category $\mathscr{C}_-$ is called an *f-functor* if it obeys the following condition:

(3.3.1) The assignment $f \in \mathrm{Mor}\,\mathscr{C}_+ \mapsto \mathscr{F}(f) \in \mathrm{Mor}\,\mathscr{C}_-$ is an f-function.

Given an f-functor $\mathscr{F}$: $\mathscr{C}_+ \to \mathscr{C}_-$ of $\mathrm{X}_\pm$-categories and $p \in \mathscr{P}(\mathrm{X}_+)$, we write $\mathscr{F}[p]$ for the functor from the category $\mathscr{C}_+[p]$ t the category $\mathscr{C}_-[p]$ induced by $\mathscr{F}$.

Given two f-functors $\mathscr{F}$, $\mathscr{G}$: $\mathscr{C}_+ \to \mathscr{C}_-$ of $\mathrm{X}_\pm$-categories, a natural f-transformation is a natural transformation α from the functor $\mathscr{F}$ to the functor $\mathscr{G}$ subject to the following condition:

(3.3.2) The assignment $x \in \mathrm{Ob}\, \mathscr{C}_+ \mapsto \alpha_x \in \mathrm{Mor}\, \mathscr{C}_-$ is an f-function.

Given two X-functors $\mathscr{F}, \mathscr{G}: \mathscr{C} \to \mathscr{D}$ of X-categories, a natural X-transformation $\alpha: \mathscr{F} \to \mathscr{G}$, and $p \in \mathbf{B}$, we write $\alpha[p]$ for the natural transformation from the functor $\mathscr{F}[p]$ to the functor $\mathscr{G}[p]$ induced by α.

Proposition 2.5.4 implies the following.

Proposition 3.3.1. Let $\mathscr{C}_\pm$ be as above. There is a bijection between the f-functors from $\mathscr{C}_+$ to $\mathscr{C}_-$ and the X_--functors from $f^*_{\mathbf{Cat}}(\mathscr{C}_+)$ to $\mathscr{C}_-$.

Proposition 3.3.2. Let $\mathscr{C}_+$ be a small_i X_+-category such that $\mathscr{C}[1_{X_+}]$ has a terminal object. Let $\mathscr{F}$ be an f-functor from $\mathscr{C}$ to a small_i X_--category $\mathscr{D}$. Then f-functor $\mathscr{F}$ maps small_i X_+-limits (X_+-finite X_+-limits, resp.) to X_--limits iff the functor $\mathscr{F}[1_X]$ preserves small_i limits (finite limits, resp.).

Proposition 3.3.3. Let $\mathscr{C}_+$ be a small_i X_+-category such that $\mathscr{C}[1_{X_+}]$ has an initial object. Let $\mathscr{F}$ be an f-functor from $\mathscr{C}$ to a small_i X_--category $\mathscr{D}$. Then f-functor $\mathscr{F}$ maps small_i X_+-colimits (X_+-finite X_+-colimits, resp.) to X_--colimits iff the functor $\mathscr{F}[1_X]$ preserves small_i colimits (finite colimits, resp.).

4. QUANTIZATION MACHINERY

Let us introduce the category to be denoted by $\mathbf{BCat}_2$. Its objects are all pairs $(X, \mathscr{A})$ of a Boolean locale X and a small_2 X-category $\mathscr{A}$. A morphism from $(X, \mathscr{A})$ to $(Y, \mathscr{B})$ in $\mathbf{BCat}_2$ is a pair $(f, \mathscr{F})$ of a morphism $f: X \to Y$ in $\mathbf{BLoc}_0$ and an f-functor $\mathscr{F}: \mathscr{B} \to \mathscr{A}$. The composition $(g, \mathscr{G}) \circ (f, \mathscr{F})$ of morphisms $(f, \mathscr{F}): (X, \mathscr{A}) \to (Y, \mathscr{B})$ and $(g, \mathscr{G}): (Y, \mathscr{B}) \to (Z, \mathscr{C})$ in $\mathbf{BCat}_2$ is defined to be $(g \circ f, \mathscr{F} \circ \mathscr{G})$. As discussed in Nishimura (1995c, §3), the category $\mathbf{BCat}_2$ has small_0 coproducts with respect to which the category $\mathbf{BCat}_2$ can and shall be put down as an orthogonal category. The assignments $(X, \mathscr{A}) \in \mathrm{Ob}\, \mathbf{BCat}_2 \mapsto X \in \mathrm{Ob}\, \mathbf{BLoc}_0$ and $(f, \mathscr{F}) \in \mathrm{Mor}\, \mathbf{BCat}_2 \mapsto f \in \mathrm{Mor}\, \mathbf{BLoc}_0$ constitute a functor to be denoted by $\Theta_{\mathbf{BLoc}}$.

We now introduce a category to be denoted by $\mathbf{BObj}_2$. Its objects are all triples $(X, \mathscr{A}, a)$ such that $(X, \mathscr{A}) \in \mathrm{Ob}\, \mathbf{BCat}_2$ and a is a total object of the X-category $\mathscr{A}$. A morphism from $(X, \mathscr{A}, a)$ to $(Y, \mathscr{B}, b)$ in $\mathbf{BObj}_2$ is a triple $(f, \mathscr{F}, \not{f})$ such that $(f, \mathscr{F})$ is a morphism from $(X, \mathscr{A})$ to $(Y, \mathscr{B})$ in the category $\mathbf{BCat}_2$ and $\not{f}$ is a total morphism from $\mathscr{F}(b)$ to a in the X-category $\mathscr{A}$. The composition $(g, \mathscr{G}, \not{g}) \circ (f, \mathscr{F}, \not{f})$ of $(f, \mathscr{F}, \not{f}): (X, \mathscr{A}, a) \to (Y, \mathscr{B}, b)$ and $(g, \mathscr{G}, \not{g}): (Y, \mathscr{B}, b) \to (Z, \mathscr{C}, c)$ in $\mathbf{BObj}_2$ is defined to be $(g \circ f, \mathscr{F} \circ \mathscr{G}, \not{f} \circ \mathscr{F}_{\not{g}})$. It is easy to see that the category $\mathbf{BObj}_2$ has small_0 coproducts with respect to which $\mathbf{BObj}_2$ can and shall be put down as an orthogonal category. The assignments $(X, \mathscr{A}, a) \in \mathrm{Ob}\, \mathbf{BObj}_2 \mapsto (X, \mathscr{A}) \in \mathrm{Ob}\, \mathbf{BCat}_2$ and $(f, \mathscr{F}, \not{f}) \in \mathrm{Ob}\, \mathbf{BObj}_2 \mapsto (f, \mathscr{F}) \in \mathrm{Mor}\, \mathbf{BCat}_2$ constitute a functor from the category $\mathbf{BObj}_2$ to the category $\mathbf{BCat}_2$ to be denoted by $\Theta_{\mathbf{BCat}}$.

Let $\mathcal{M}$ be a manual of Boolean locales, which shall be fixed throughout the rest of this section. An *empirical framework over* $\mathcal{M}$ is a functor Φ from $\mathcal{M}$ to $\mathbf{BCat}_2$ subject to the following conditions:

(4.1) It maps orthogonal $\mathcal{M}$-sum diagrams to orthogonal sum diagrams in $\mathbf{BCat}_2$.

(4.2) $\Theta_{\mathbf{BLoc}} \circ \Phi$ is the identity functor of $\mathcal{M}$ into $\mathbf{BLoc}_0$.

For an empirical framework Φ over $\mathcal{M}$, we denote by $\Phi_{\mathcal{C}at}$ the function with the same domain of Φ such that $\Phi(X) = (X, \Phi_{\mathcal{C}at}(X))$ for each $X \in \mathrm{Ob}\mathcal{M}$ and $\Phi(f) = (f, \Phi_{\mathcal{C}at}(f))$ for each $f \in \mathrm{Mor}\mathcal{M}$.

Given an empirical framework Φ over $\mathcal{M}$, we now introduce a category to be denoted by $\mathbf{EObj}(\Phi)$. Its objects are all functors $\mathfrak{F}$ from $\mathcal{M}$ to $\mathbf{BObj}_2$ abiding by the following conditions:

(4.3) It maps orthogonal $\mathcal{M}$-sum diagrams in $\mathcal{M}$ to orthogonal sum diagrams in $\mathbf{BObj}_2$.

(4.4) $\Theta_{\mathbf{BCat}} \circ \mathfrak{F} = \Phi$.

Given such a functor $\mathfrak{F}: \mathcal{M} \to \mathbf{BObj}_2$, we denote by $\mathfrak{F}_{\mathcal{O}bj}$ the function with the same domain of $\mathfrak{F}$ such that the value of $\mathfrak{F}_{\mathcal{O}bj}(?)$ is the third component of the triple $\mathfrak{F}(?)$. A morphism from $\mathfrak{F}$ to $\mathfrak{G}$ in $\mathbf{EObj}(\Phi)$ is an assignment ζ to each $X \in \mathrm{Ob}\mathcal{M}$ of a total morphism $\zeta_X: \mathfrak{F}_{\mathcal{O}bj}(X) \to \mathfrak{F}_{\mathcal{O}bj}(Y)$ satisfying the following condition:

(4.5) The diagram

$$\begin{array}{ccc}
\Phi_{\mathcal{C}at}(f)(\mathfrak{F}_{\mathcal{O}bj}(Y)) & \xrightarrow{\mathfrak{F}_{\mathcal{O}bj}(f)} & \mathfrak{F}_{\mathcal{O}bj}(X) \\
{\scriptstyle \Phi_{\mathcal{C}at}(f)(\zeta_Y)} \downarrow & & \downarrow {\scriptstyle \zeta_X} \\
\Phi_{\mathcal{C}at}(f)(\mathfrak{G}_{\mathcal{O}bj}(Y)) & \xrightarrow[\mathfrak{G}_{\mathcal{O}bj}(f)]{} & \mathfrak{G}_{\mathcal{O}bj}(X)
\end{array}$$

is commutative for every $f: X \to Y \in \mathrm{Mor}\mathcal{M}$.

The composition $\eta \circ \zeta$ of morphisms $\zeta: \mathfrak{F} \to \mathfrak{G}$ and $\eta: \mathfrak{G} \to \mathfrak{H}$ in $\mathbf{EOBj}(\Phi)$ is defined to be the assignment $X \in \mathrm{Ob}\mathcal{M} \mapsto \eta_X \circ \zeta_X$.

5. EMPIRICAL TOPOS THEORY

5.1. Booleanization

Let X be a Boolean locale, which shall be fixed throughout this subsection. By interpreting the notion of a small_2 topos in the topos $\mathbf{BEns}_2(X)$, we get the notion of a (small$_2$) X-topos, for which it is easy to see the following result:

Proposition 5.1.1. A small$_2$ X-category $\mathscr{E}$ is an X-topos iff $\mathscr{E}[1_X]$ is a topos.

By interpreting the notion of a geometric functor of small$_2$ toposes within the topos $\mathbf{BEns}_2(X)$, we get the notion of a geometric X-functor of small$_2$ X-toposes, for which it is easy to see the following.

Proposition 5.1.2. Given small$_2$ X-toposes $\mathscr{E}_\pm$, an X-functor $\mathscr{E}$ from the X-category $\mathscr{E}_+$ to the X-category $\mathscr{E}_-$ is a geometric X-functor from the X-topos $\mathscr{E}_+$ to the X-topos $\mathscr{E}_-$ iff $\mathscr{E}[1_X]$ is a geometric functor from the topos $\mathscr{E}_+[1_X]$ to the topos $\mathscr{E}_-[1_X]$.

Proof. This follows from Proposition 3.2.3 and 3.2.5. ■

By interpreting the notion of a topology on a small$_2$ topos within the topos $\mathbf{BEns}_2(X)$, we get the notion of an X-*topology* on a small$_2$ X-topos $\mathscr{E}$, which is a topology on the topos $\mathscr{E}[1_X]$. A (*small*$_2$) *localized* X-*topos* is a pair $(\mathscr{E}, j)$ of a small$_2$ X-topos $\mathscr{E}$ and an X-topology j on $\mathscr{E}$. Given a localized X-topos $(\mathscr{E}, j)$, its associated X-topos $\mathscr{BSh}(X; \mathscr{E}, j)$ can be obtained simply by interpreting $\mathbf{Sh}(\mathscr{E}, j)$ within the topos $\mathbf{BEns}_2(X)$ and the left X-adjoint of the inclusion X-functor i_j of $\mathscr{BSh}(X; \mathscr{E}, j)$ into $\mathscr{E}$ is denoted by a_j.

Given an X-topos $\mathscr{E}$, by interpreting the notions of $\mathbf{Cat}(\mathscr{E})$, $\mathbf{Rng}(\mathscr{E})$, and $\mathbf{LocRng}(\mathscr{E})$ within the topos $\mathbf{BEns}_2(X)$, we get the notions of $\mathscr{BCat}(\mathscr{E})$, $\mathscr{BRng}(\mathscr{E})$, and $\mathscr{BLocRng}(\mathscr{E})$ externally. By interpreting the notion of $\mathbf{Ens}_i$ within the topos $\mathbf{BEns}_2(X)$ ($i = 0, 1, 2$), we get the external notion of $\mathscr{BEns}_i(X)$, which is an X-topos. By interpreting the notions of $\mathbf{Cat}_i$, $\mathbf{Rng}_i$, $\mathbf{LocRng}_i$, and $\mathbf{fp\text{-}Rng}_i$ within the topos $\mathbf{BEns}_2(X)$, we get the notions of $\mathscr{BCat}_i(X)$, $\mathscr{BRng}_i(X)$, $\mathscr{BLocRng}_i(X)$, and $fp\text{-}\mathscr{BRng}_i(X)$ externally.

5.2. Relations Between Two Booleanizations

Let $\mathfrak{f}: X_- \to X_+$ be a morphism of $\mathbf{BLoc}$, which shall be fixed throughout this subsection. Given small$_2$ $X_\pm$-toposes $\mathscr{E}_\pm$, a *geometric* $\mathfrak{f}$-functor from the X_+-topos $\mathscr{E}_+$ to the X_--topos $\mathscr{E}_-$ is an $\mathfrak{f}$-functor $\mathscr{E}$ from the X_+-category $\mathscr{E}_+$ to the X_--category $\mathscr{E}_-$ subject to the following condition:

(5.2.1) $\mathscr{E}[1_{X_+}]$ is a geometric functor from the topos $\mathscr{E}_+[1_{X_+}]$ to the topos $\mathscr{E}_-[1_{X_-}]$.

Given two geometric $\mathfrak{f}$-functors $\mathscr{E}: \mathscr{E}_+ \to \mathscr{E}_-$ and $\mathscr{F}: \mathscr{F}_+ \to \mathscr{F}_-$, a geometric $\mathfrak{f}$-*conjugation* from the geometric $\mathfrak{f}$-functor $\mathscr{E}$ to the geometric $\mathfrak{f}$-functor $\mathscr{F}$ is a triple $(\mathscr{H}_-, \mathscr{H}_+, \alpha)$ of two geometric $X_\pm$-functors $\mathscr{H}_\pm: \mathscr{E}_\pm \to \mathscr{F}_\pm$ and a natural $\mathfrak{f}$-transformation $\alpha: \mathscr{F} \circ \mathscr{H}_+ \to \mathscr{H}_- \circ \mathscr{E}$ subject to the following condition:

(5.2.2)$(\mathscr{H}_-[1_{X_-}], \mathscr{H}_+[1_{X_+}], \alpha[1_{X_+}])$ is a geometric conjugation from the geometric functor $\mathscr{E}[1_{X_+}]: \mathscr{E}_+[1_{X_+}] \to \mathscr{E}_-[1_{X_-}]$ to the geometric functor $\mathscr{F}[1_{X_+}]: \mathscr{F}_+[1_{X_+}] \to \mathscr{F}_-[1_{X_-}]$.

Given two geometric f-conjugations $(\mathscr{H}_-, \mathscr{H}_+, \alpha)$ and $(\mathscr{K}_-, \mathscr{K}_+, \beta)$ from a geometric f-functor $\mathscr{E}: \mathscr{E}_+ \to \mathscr{E}_-$ to a geometric f-functor $\mathscr{F}: \mathscr{F}_+ \to \mathscr{F}_-$, a *geometric* f-transformation from $(\mathscr{H}_-, \mathscr{H}_+, \alpha)$ to $(\mathscr{K}_-, \mathscr{K}_+, \beta)$ is a pair (σ_-, σ_+) of natural $X_\pm$-transformations $\sigma_\pm: \mathscr{H}_\pm \to \mathscr{K}_\pm$ subject to the following condition:

(5.2.3) $(\sigma_-[1_{X_-}], \sigma_+[1_{X_+}])$ is a geometric transformation from the geometric conjugation

$$(\mathscr{H}_-[1_{X_-}], \mathscr{H}_+[1_{X_+}], \alpha[1_{X_+}]): \mathscr{E}[1_{X_+}] \to \mathscr{F}[1_{X_+}]$$

to the geometric conjugation

$$(\mathscr{K}_-[1_{X_-}], \mathscr{K}_+[1_{X_+}], \beta[1_{X_+}]): \mathscr{E}[1_{X_+}] \to \mathscr{F}[1_{X_+}]$$

Given small$_2$ localized $X_\pm$-toposes $(\mathscr{E}_\pm, j_\pm)$, a geometric f-functor $\mathscr{E}$: $\mathscr{E}_+ \to \mathscr{E}_-$ is said to be *localized* with respect to $j_\pm$ if the geometric functor $\mathscr{E}[1_{X_+}]: \mathscr{E}_+[1_{X_+}] \to \mathscr{E}_-[1_{X_-}]$ is localized with respect to $j_\pm$. In this case the geometric f-functor $\mathscr{E}$ naturally induces a geometric f-functor

$$\mathscr{BSh}^*(\mathfrak{f}; \mathscr{E}, j_-, j_+): \mathscr{BSh}(X_+; \mathscr{E}_+, j_+) \to \mathscr{BSh}(X_-; \mathscr{E}_-, j_-)$$

by $a_{j_-} \circ \mathscr{E} \circ i_{j_+}$.

A geometric f-functor $\mathscr{E}: \mathscr{E}_+ \to \mathscr{E}_-$ of $X_\pm$-toposes naturally gives rise to f-functors

$$\mathscr{E}_{\mathscr{BCat}}: \mathscr{BCat}(\mathscr{E}_+) \to \mathscr{BCat}(\mathscr{E}_-)$$

$$\mathscr{E}_{\mathscr{BRng}}: \mathscr{BRng}(\mathscr{E}_+) \to \mathscr{BRng}(\mathscr{E}_-)$$

$$\mathscr{E}_{\mathscr{BLocRng}}: \mathscr{BLocRng}(\mathscr{E}_+) \to \mathscr{BLocRng}(\mathscr{E}_-)$$

The morphism $\mathfrak{f}: X_- \to X_+$ of Boolean locales gives rise to a geometric f-functor $\mathfrak{f}^*_{\mathscr{BEns}_i}: \mathscr{BEns}_i(X_+) \to \mathscr{BEns}_i(X_-)$, which then induces f-functors

$$\mathfrak{f}^*_{\mathscr{BCat}_i}: \mathscr{BCat}_i(X_+) \to \mathscr{BCat}_i(X_-)$$

$$\mathfrak{f}^*_{\mathscr{BRng}_i}: \mathscr{BRng}_i(X_+) \to \mathscr{BRng}_i(X_-)$$

$$\mathfrak{f}^*_{\mathscr{BLocRng}_i}: \mathscr{BLocRng}_i(X_+) \to \mathscr{BLocRng}_i(X_-)$$

$$\mathfrak{f}^*_{fp\text{-}\mathscr{Brng}_i}: fp\text{-}\mathscr{BRng}_i(X_+) \to fp\text{-}\mathscr{BRns}_i(X_-)$$

Proposition 2.5.4 implies the following:

Proposition 5.2.1. Let $\mathscr{C}_\pm$ be small$_2$ $X_\pm$-categories. Then there is a bijection between the f-functors from $\mathscr{C}_+$ to $\mathscr{C}_-$ and the X_--functors from $\mathfrak{f}^*_{\mathscr{BCat}_2}(\mathscr{C}_+)$ to $\mathscr{C}_-$.

5.3. Quantization

Let $\mathcal{M}$ be a manual of Boolean locales, which shall be fixed throughout this subsection. A *(small$_2$)$\mathcal{M}$-topos* is a function $\mathfrak{T}$ on Ob$\mathcal{M}$ ∪ Mor$\mathcal{M}$ subject to the following conditions:

(5.3.1) $\mathfrak{T}(X)$ is a small$_2$ X-topos for each X ∈ Ob$\mathcal{M}$.
(5.3.2) $\mathfrak{T}(f)$ is a geometric f-functor from $\mathfrak{T}(Y)$ to $\mathfrak{T}(X)$ for each f: X → Y ∈ Mor$\mathcal{M}$.
(5.3.3) The assignments X ∈ Ob$\mathcal{M}$ ↦ (X, $\mathfrak{T}(X)$) and f ∈ Mor$\mathcal{M}$ ↦ (f, $\mathfrak{T}(f)$) constitute an empirical framework over $\mathcal{M}$.

A small$_2$$\mathcal{M}$-topos $\mathfrak{T}$ is said to be *small$_1$-$\mathcal{M}$-cocomplete* if $\mathfrak{T}(X)$ is small$_1$-cocomplete for any X ∈ Ob$\mathcal{M}$.

Given two $\mathcal{M}$-toposes $\mathfrak{T}_\pm$, a *geometric $\mathcal{M}$-functor* from $\mathfrak{T}_+$ to $\mathfrak{T}_-$ is a pair ($\mathfrak{F}$, α) of an assignment $\mathfrak{F}$ of a geometric X-functor $\mathfrak{T}(X)$: $\mathfrak{T}_+(X) \to \mathfrak{T}_-(X)$ to each X ∈ Ob$\mathcal{M}$ and a natural f-transformation α(f): $\mathfrak{T}_-(f) \circ \mathfrak{F}(Y) \to \mathfrak{F}(X) \circ \mathfrak{T}_+(f)$ to each f: X → Y ∈ Mor$\mathcal{M}$ abiding by the following condition:

(5.3.4) For each f: X → Y ∈ Mor$\mathcal{M}$, the triple ($\mathfrak{F}(X)$, $\mathfrak{F}(Y)$, α(f)) is a geometric f-conjugation from the geometric f-functor $\mathfrak{T}_+(f)$: $\mathfrak{T}_+(Y) \to \mathfrak{T}_+(X)$ to the geometric f-functor $\mathfrak{T}_-(f)$: $\mathfrak{T}_-(Y) \to \mathfrak{T}_-(X)$.

Given two $\mathcal{M}$-toposes $\mathfrak{T}_\pm$ and two geometric $\mathcal{M}$-functors ($\mathfrak{F}$, α) and ($\mathfrak{G}$, β) from $\mathfrak{T}_+$ to $\mathfrak{T}_-$, a *natural $\mathcal{M}$-transformation* from ($\mathfrak{F}$, α) to ($\mathfrak{G}$, β) is an assignment of a natural X-transformation σ(X): $\mathfrak{F}(X) \to \mathfrak{G}(X)$ to each X ∈ Ob$\mathcal{M}$ subject to the following condition:

(5.3.5) For each f: X → Y ∈ Mor$\mathcal{M}$, (σ(X), σ(Y)) is a geometric f-transformation from the geometric f-conjugation ($\mathfrak{F}(X)$, $\mathfrak{F}(Y)$, α(f)): $\mathfrak{T}_+(f) \to \mathfrak{T}_-(f)$ to the geometric f-conjugation ($\mathfrak{G}(X)$, $\mathfrak{G}(Y)$, β(f)): $\mathfrak{T}_+(f) \to \mathfrak{T}_-(f)$.

Given an $\mathcal{M}$-topos $\mathfrak{T}$, an *$\mathcal{M}$-topology* on $\mathfrak{T}$ is an assignment j of an $\mathcal{M}$(X)-topology j(X) on the X-topos $\mathfrak{T}(X)$ to each X ∈ Ob$\mathcal{M}$ subject to the following condition:

(5.3.6) For each f: X → Y ∈ Mor$\mathcal{M}$, the geometric f-functor $\mathfrak{T}(f)$: $\mathfrak{T}(Y) \to \mathfrak{T}(X)$ is localized with respect to j(Y) and j(X).

A *localized $\mathcal{M}$-topos* is a pair ($\mathfrak{T}$, j) of an $\mathcal{M}$-topos $\mathfrak{T}$ and an $\mathcal{M}$-topology j on $\mathfrak{T}$ whose associated $\mathcal{M}$-topos $\mathfrak{ESh}(\mathcal{M}; \mathfrak{T}, j)$ is defined as follows:

(5.3.7) $\mathfrak{ESh}(\mathcal{M}; \mathfrak{T}, j)(X) = \mathcal{BSh}(X; \mathfrak{T}(X), j(X))$ for each X ∈ Ob$\mathcal{M}$.
(5.3.8) $\mathfrak{ESh}(\mathcal{M}; \mathfrak{T}, j)(f) = \mathcal{BSh}^*(f; \mathfrak{T}(f), j(X), j(Y))$ for each f: X → Y ∈ Mor$\mathcal{M}$.

Given a localized $\mathcal{M}$-topos ($\mathfrak{T}$, j), its associated geometric $\mathcal{M}$-functor ($\mathbf{a}_j$, α_j): $\mathfrak{T} \to \mathfrak{ESh}(\mathcal{M}; \mathfrak{T}, j)$ is defined as follows:

(5.3.9) For each $X \in \mathrm{Ob}\,\mathscr{M}$, $a_j(X)$ shall be the geometric X-functor $a_{j(X)}$: $\mathfrak{X}(X) \to \mathscr{BSh}(X; \mathfrak{X}(X), j(X))$ associated with the localized X-topos $(\mathfrak{X}(X), j(X))$.

(5.3.10) For each $f: X \to Y \in \mathrm{Ob}\,\mathscr{M}$, $\alpha_j(f)$ shall be the identity f-transformation.

6. EMPIRICAL GROTHENDIECK TOPOSES

6.1. Booleanization

Let X be a Boolean locale, which shall be fixed throughout this subsection. By interpreting the notion of a Grothendieck topology on a small_1 category in the topos $\mathbf{BEns}_1(X)$, we get the notion of a *Grothendieck* X-*topology* on a small_1 X-category $\mathscr{C}$. A pair $(\mathscr{C}, \mathscr{L})$ of a small_1 X-category $\mathscr{C}$ and a Grothendieck X-topology $\mathscr{L}$ on $\mathscr{C}$ is called a (small_1) X-*site*. Given a small_1 X-category $\mathscr{C}$, by interpreting the notion of $\mathbf{PreSh}_1(\mathscr{C})$ within the topos $\mathbf{BEns}_1(X)$, we get the notion of $\mathscr{BPreSh}_1(X; \mathscr{C})$ externally, and the Yoneda embedding of $\mathscr{C}$ into $\mathscr{BPreSh}_1(X; \mathscr{C})$ within the topos $\mathbf{BEns}_1(X)$ is denoted by $\mathscr{y}$.

By simply Booleanizing Theorem 2.3.1, we have the following.

Theorem 6.1.1. Given a small_1 X-category $\mathscr{C}$, there is a bijection between the Grothendieck X-topologies on $\mathscr{C}$ and the X-topologies on the X-topos $\mathscr{BPreSh}_1(X; \mathscr{C})$.

Given a Grothendieck X-topology $\mathscr{L}$ on a small_1 X-category $\mathscr{C}$, the X-topology on $\mathscr{BPreSh}_1(X; \mathscr{C})$ corresponding to $\mathscr{L}$ is denoted by $\mathscr{j}[\mathscr{L}]$.

6.2. Relations Between Two Booleanizations

Let $f: X_- \to X_+$ be a morphism of **BLoc**, which shall be fixed throughout this subsection. For the proofs of the following three theorems, the reader is referred to Nishimura (1996a, §2.5).

Theorem 6.2.1. Let φ be an f-functor from a small_1 X_+-category $\mathscr{C}_+$ to a small_1 X_--category $\mathscr{C}_-$. Then there is, up to natural f-isomorphisms, a unique f-functor

$$\mathscr{BPreSh}_1^*(f; \varphi): \mathscr{BPreSh}_1(X_+; \mathscr{C}_+) \to \mathscr{BPreSh}_1(X_-; \mathscr{C}_-)$$

mapping small$_1$ X_+-colimits to X_--colimits and making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{BPreSh}_1(X_+;\mathscr{C}_+) & \xrightarrow{\mathcal{BPreSh}_1^*(f;\,\varphi)} & \mathcal{BPreSh}_1(X_-;\mathscr{C}_-) \\ \uparrow{\scriptstyle \mathcal{Y}} & & \uparrow{\scriptstyle \mathcal{Y}} \\ \mathscr{C}_+ & \xrightarrow[\varphi]{} & \mathscr{C}_- \end{array}$$

Theorem 6.2.2. In the above theorem, if we assume also that $\mathscr{C}_+$ is X_+-finitely X_+-complete and that $\mathscr{F}$ maps X_+-finite X_+-limits to X_--limits, then the f-functor

$$\mathcal{BPreSh}_1^*(f;\,\varphi)\colon \mathcal{BPreSh}_1(X_+;\,\mathscr{C}_+) \to \mathcal{BPreSh}_1(X_-;\,\mathscr{C}_-)$$

is a geometric f-functor.

Given two small$_1$ $X_\pm$-sites $\mathscr{C}_\pm, \mathscr{L}_\pm)$ with $\mathscr{C}_+[1_{X_+}]$ being nonempty, an f-functor $\varphi\colon \mathscr{C}_+ \to \mathscr{C}_-$ is said to be an *f-morphism* from the X_+-site $(\mathscr{C}_+,\mathscr{L}_+)$ to the X_--site $(\mathscr{C}_-, \mathscr{L}_-)$ if $\varphi[1_{X_+}]$ is a morphism of sites from $(\mathscr{C}_+[1_{X_+}],\mathscr{L}_+[1_{X_+}])$ to $(\mathscr{C}_-[1_{X_-}],\mathscr{L}_-[1_{X_-}])$.

Theorem 6.2.3. For any f-morphism φ from a small$_1$ X_+-site $(\mathscr{C}_+,\mathscr{L}_+)$ to a small$_1$ X_--site $(\mathscr{C}_-, \mathscr{L}_-)$ with $\mathscr{C}_+[1_{X_+}]$ being nonempty, the geometric f-functor

$$\mathcal{BPreSh}_1^*(f;\,\varphi)\colon \mathcal{BPreSh}_1(X_+;\,\mathscr{C}_+) \to \mathcal{BPreSh}_1(X_-;\,\mathscr{C}_-)$$

is localized with respect to the Grothendieck $X_\pm$-topologies$\mathcal{J}[\mathscr{L}_\pm]$.

6.3. Quantization

Let $\mathscr{M}$ be a manual of Boolean locales, which shall be fixed throughout this subsection. A *(small$_2$)$\mathscr{M}$-category* $\mathfrak{C}$ is a function $\mathfrak{C}$ on $\mathrm{Ob}\mathscr{M} \cup \mathrm{Mor}\mathscr{M}$ subject to the following conditions:

(6.3.1) For any $X \in \mathrm{Ob}\mathscr{M}$, $\mathfrak{C}(X)$ is a small$_2$ X-category.
(6.3.2) For any $f\colon X \to Y \in \mathrm{Mor}\mathscr{M}$, $\mathfrak{C}(f)$ is an f-functor from $\mathfrak{C}(Y)$ to $\mathfrak{C}(X)$.
(6.3.3) The assignments $X \in \mathrm{Ob}\mathscr{M} \mapsto (X, \mathfrak{C}(X))$ and $f \in \mathrm{Mor}\mathscr{M} \mapsto (f, \mathfrak{C}(f))$ constitute an empirical framework over $\mathscr{M}$.

The condition (6.3.3) gives a natural bijection between the $\mathscr{M}$-categories and the empirical frameworks over $\mathscr{M}$, so that we can safely speak, e.g., of the category **EObj**($\mathfrak{C}$) of empirical objects of $\mathfrak{C}$.

An $\mathscr{M}$-category $\mathfrak{C}$ is said to be *$\mathscr{M}$-finitely $\mathscr{M}$-complete* if it obeys the following conditions:

(6.3.4) For any X ∈ Ob $\mathscr{M}$, the X-category $\mathfrak{C}$(X) is X-finitely X-complete.
(6.3.5) For any f: X → Y ∈ Mor $\mathscr{M}$, the f-functor $\mathfrak{C}$(f) maps Y-finite Y-limits to X-limits.

Given a small$_1$ $\mathscr{M}$-finitely $\mathscr{M}$-complete $\mathscr{M}$-category $\mathfrak{C}$, its associated $\mathscr{M}$-topos $\mathfrak{EBreSh}_1$($\mathscr{M}$; $\mathfrak{C}$) is defined as follows:

(6.3.6) For any X ∈ Ob $\mathscr{M}$, $\mathfrak{EBreSh}_1$($\mathscr{M}$ $\mathfrak{C}$)(X) shall be the X-topos $\mathscr{BPreSh}_1$(X; $\mathfrak{C}$(X)).
(6.3.7) For any f: X → Y ∈ Mor $\mathscr{M}$, $\mathfrak{EBreSh}_1$($\mathscr{M}$; $\mathfrak{C}$)(f) shall be the geometric f-functor $\mathscr{BPreSh}_1$(f; $\mathfrak{C}$(f)) from $\mathscr{BPreSh}_1$(Y; $\mathfrak{C}$(Y)) to $\mathscr{BPreSh}_1$(X; $\mathfrak{C}$(X)).

Given a small$_1$ $\mathscr{M}$-finitely $\mathscr{M}$-complete $\mathscr{M}$-category $\mathfrak{C}$, a *Grothendieck $\mathscr{M}$-topology* on $\mathfrak{C}$ is an assignment $\mathfrak{L}$ of a Grothendieck X-topology $\mathfrak{L}$(X) on the small$_1$ X-finitely X-complete X-category $\mathfrak{C}$(X) to each X ∈ Ob $\mathscr{M}$ subject to the following condition:

(6.3.8) For each f: X → Y ∈ Mor $\mathscr{M}$, the f-functor $\mathfrak{C}$(f): $\mathfrak{C}$(Y) → $\mathfrak{C}$(X) is an f-morphism from the Y-site ($\mathfrak{C}$(Y), $\mathfrak{L}$(Y)) to the X-site ($\mathfrak{C}$(X), $\mathfrak{L}$(X)).

A pair ($\mathfrak{C}$, $\mathfrak{L}$) of a small$_1$ $\mathscr{M}$-finitely $\mathscr{M}$-complete $\mathscr{M}$-category and a Grothendieck $\mathscr{M}$-topology on $\mathfrak{C}$ is called a *(small$_1$) $\mathscr{M}$-site*. Given a small$_1$ $\mathscr{M}$-site ($\mathfrak{C}$, $\mathfrak{L}$), the assignment X ∈ Ob $\mathscr{M}$ $\mapsto_{\mathscr{J}}$[$\mathfrak{L}$(X)], to be denoted j[$\mathfrak{L}$], is an $\mathscr{M}$-topology on the $\mathscr{M}$-topos $\mathfrak{EBreSh}_1$($\mathscr{M}$; $\mathfrak{C}$). The resulting $\mathscr{M}$-topos $\mathfrak{ESh}$($\mathscr{M}$; $\mathfrak{EBreSh}_1$($\mathscr{M}$; $\mathfrak{C}$), j[$\mathfrak{L}$]) is denoted by $\mathfrak{ESh}_1$($\mathscr{M}$; $\mathfrak{C}$, $\mathfrak{L}$).

7. THE 2-CATEGORY OF EMPIRICAL TOPOSES

7.1. The 2-Category BTOP

The 2-category $\mathbf{BTOP}_2$ consists of the following entities:

(7.1.1) Its objects are all pairs (X, $\mathscr{T}$) of X ∈ Ob $\mathbf{BLoc}_0$ and a small$_2$ X-topos.
(7.1.2) Its morphisms from (X; $\mathscr{T}$) to (Y, $\mathscr{S}$) are all pairs (f, $\mathscr{F}$) of f: X → Y ∈ Mor $\mathbf{BLoc}_0$ and a geometric f-functor $\mathscr{F}$: $\mathscr{S}$ → $\mathscr{T}$.
(7.1.3) Its 2-arrows from (f, $\mathscr{F}$): (X, $\mathscr{T}$) → (Y, $\mathscr{S}$) to (g, $\mathscr{G}$): (X, $\mathscr{T}$) → (Y, $\mathscr{S}$) exist provided that

$$(7.1.3.1)\quad f = g$$

and are all natural f-transformations from $\mathscr{F}$ to $\mathscr{G}$.

It is easy to see the following:

Theorem 7.1.1. **BTOP**$_2$ is a 2-category with respect to the following operations:

(7.1.4) The composite of morphisms $(f, \mathcal{F})$: $(X, \mathcal{S}) \to (Y, \mathcal{T})$ and $(g, \mathcal{G})$: $(Y, \mathcal{T}) \to (Z, \mathcal{U})$ shall be $(g \circ f, \mathcal{F} \circ \mathcal{G})$, to be denoted by $(g, \mathcal{G}) \circ (f, \mathcal{F})$.

(7.1.5) The vertical composite of a 2-arrow α from $(f, \mathcal{F})$: $(X, \mathcal{S}) \to (Y, \mathcal{T})$ to $(g, \mathcal{G})$: $(X, \mathcal{S}) \to (Y, \mathcal{T})$ and a 2-arrow β from $(g, \mathcal{G})$: $(X, \mathcal{S}) \to (Y, \mathcal{T})$ to $(h, \mathcal{H})$: $(X, \mathcal{S}) \to (Y, \mathcal{T})$ shall be $\beta \cdot \alpha$.

(7.1.6) The horizontal composite of a 2-arrow α from $(f, \mathcal{F})$: $(X, \mathcal{S}) \to (Y, \mathcal{T})$ to $(g, \mathcal{G})$: $(X, \mathcal{S}) \to (Y, \mathcal{T})$ and a 2-arrow β from $(f', \mathcal{F}')$: $(Y, \mathcal{S}) \to (Z, \mathcal{U}$ to $(g', \mathcal{G}')$: $(Y, \mathcal{T}) \to (Z, \mathcal{U})$ shall be $\alpha \circ \beta$.

The objects and morphisms of **BTOP**$_2$ constitute a category to be denoted by **BTop**$_2$.

7.2. The 2-Categories BGEOM* and BGEOM$_*$

The 2-category **BGEOM**$_2^*$ consists of the following entities:

(7.2.1) Its objects are all pairs $(f, \mathcal{F})$ of $f: X_- \to X_+ \in$ Mor **BLoc**$_0$ and a geometric f-functor $\mathcal{F}$ from a small$_2$ X_+-topos $\mathcal{E}_+$ to a small$_2$ X_--topos $\mathcal{E}_-$.

(7.2.2) Its morphisms from $(f: X_- \to X_+, \mathcal{F}: \mathcal{E}_+ \to \mathcal{E}_-)$ to $(f': X'_- \to X'_+, \mathcal{F}': \mathcal{E}'_+ \to \mathcal{E}'_-)$ are all triples $((h_-: X_- \to X'_-, \mathcal{H}_-: \mathcal{E}'_- \to \mathcal{E}_-), (h_+: X_+ \to X'_+, \mathcal{H}_+: \mathcal{E}'_+ \to \mathcal{E}_+), \alpha)$, where

(7.2.2.1) $(h_\pm: X_\pm \to X'_\pm, \mathcal{H}_\pm: \mathcal{E}'_\pm \to \mathcal{E}_\pm) \in$ Ob **BGEOM**$_2^*$
(7.2.2.2) $h_+ \circ f = f' \circ h_-$
(7.2.2.3) α is a natural $h_+ \circ f$-transformation from $\mathcal{F} \circ \mathcal{H}_+$ to $\mathcal{H}_- \circ \mathcal{F}'$.

(7.2.3) Its 2-arrows from $((h_-, \mathcal{H}_-), (h_+, \mathcal{H}_+), \alpha)$: $(f, \mathcal{F}) \to (f', \mathcal{F}')$ to $((k_-, \mathcal{K}_-), (k_+, \mathcal{K}_+), \beta)$: $(f, \mathcal{F}) \to (f', \mathcal{F}')$ exist provided that

(7.2.3.1) $h_\pm = k_\pm$

and are all pairs (σ_-, σ_+), where

(7.2.3.2) $\sigma_\pm$ are natural $h_\pm$-transformations from $\mathcal{H}_\pm$ to $\mathcal{K}_\pm$.
(7.2.3.3) $\beta \cdot (\mathcal{F} \circ \sigma_+) = (\sigma_- \circ \mathcal{F}') \cdot \alpha$

By the same token as in Section 2.1, we have the following result.

Theorem 7.2.1. **BGEOM**$_2^*$ is a 2-category with respect to the following operations:

(7.2.4) Given two morphisms

$$((\mathbf{h}_-,\mathscr{H}_-), (\mathbf{h}_+,\mathscr{H}_+), \alpha): (\mathbf{f},\mathscr{F}) \to (\mathbf{f}',\mathscr{F}')$$

$$((\mathbf{k}_-,\mathscr{K}_-), (\mathbf{k}_+,\mathscr{K}_+), \beta): (\mathbf{f}',\mathscr{F}') \to (\mathbf{f}'',\mathscr{F}'')$$

then

$$((\mathbf{k}_-,\mathscr{K}_-), (\mathbf{k}_+,\mathscr{K}_+), \beta) \circ ((\mathbf{h}_-,\mathscr{H}_-), (\mathbf{h}_+,\mathscr{H}_+), \alpha)$$

shall be

$$((\mathbf{k}_- \circ \mathbf{h}_-,\mathscr{H}_- \circ \mathscr{K}_-), (\mathbf{k}_+ \circ \mathbf{h}_+,\mathscr{H}_+ \circ \mathscr{K}_+), (\mathscr{H}_- \circ \beta)\cdot(\alpha \circ \mathscr{K}_+))$$

(7.2.5) Given three parallel morphisms $((\mathbf{h}_-,\mathscr{H}_-), (\mathbf{h}_+,\mathscr{H}_+), \alpha)$, $((\mathbf{k}_-,\mathscr{K}_-), (\mathbf{k}_+,\mathscr{K}_+), \beta)$, and $((\mathbf{l}_-,\mathscr{L}_-), (\mathbf{l}_+,\mathscr{L}_+), \gamma)$: $(\mathbf{f},\mathscr{F}) \to (\mathbf{f}',\mathscr{F}')$, the vertical composite of 2-arrows

$$(\sigma_-, \sigma_+): ((\mathbf{h}_-,\mathscr{H}_-), (\mathbf{h}_+,\mathscr{H}_+), \alpha) \to ((\mathbf{k}_-,\mathscr{K}_-), (\mathbf{k}_+,\mathscr{K}_+), \beta)$$

and

$$(\tau_-, \tau_+): ((\mathbf{k}_-,\mathscr{K}_-), (\mathbf{k}_+,\mathscr{K}_+), \beta) \to ((\mathbf{l}_-,\mathscr{L}_-), (\mathbf{l}_+,\mathscr{L}_+), \gamma)$$

in notation $(\tau_-, \tau_+)\cdot(\sigma_-, \sigma_+)$, shall be $(\tau_-\cdot\sigma_-, \tau_+\cdot\sigma_+)$.

(7.2.6) Given two pairs of parallel morphisms

$$((\mathbf{h}_-,\mathscr{H}_-), (\mathbf{h}_+,\mathscr{H}_+), \alpha), ((\mathbf{k}_-,\mathscr{K}_-), (\mathbf{k}_+,\mathscr{K}_+), \beta): (\mathbf{f},\mathscr{F}) \to (\mathbf{f}',\mathscr{F}')$$

$$((\mathbf{h}'_-,\mathscr{H}'_-), (\mathbf{h}'_+,\mathscr{H}'_+), \alpha'), ((\mathbf{k}'_-,\mathscr{K}'_-), (\mathbf{k}_+,\mathscr{K}'_+), \beta'): (\mathbf{f}',\mathscr{F}') \to (\mathbf{f}'',\mathscr{F}'')$$

the horizontal composite of 2-arrows

$$(\sigma_-, \sigma_+): ((\mathbf{h}_-,\mathscr{H}_-), (\mathbf{h}_+,\mathscr{H}_+), \alpha) \to ((\mathbf{k}_-,\mathscr{K}_-), (\mathbf{k}_+,\mathscr{K}_+), \beta)$$

$$(\tau_-, \tau_+): ((\mathbf{h}'_-,\mathscr{H}'_-), (\mathbf{h}'_+,\mathscr{H}'_+), \alpha') \to ((\mathbf{k}'_-,\mathscr{K}'_-), (\mathbf{k}'_+,\mathscr{K}'_+), \beta')$$

in notation $(\tau_-, \tau_+) \circ (\sigma_-, \sigma_+)$, shall be $(\sigma_- \circ \tau_-, \sigma_+ \circ \tau_+)$.

The objects and morphisms of $\mathbf{BGEOM}_2^*$ constitute a category which we denote by $\mathbf{BGeom}_2^*$.

Now we introduce a variant of the 2-category $\mathbf{BGEOM}_2^*$, to be denoted by $\mathbf{BGEOM}_{*}^{2}$, which consists of the following entities:

(7.2.7) Its objects are all pairs $(\mathbf{f},\mathscr{F})$ of $\mathbf{f}$: $\mathbf{X}_- \to \mathbf{X}_+ \in \operatorname{Mor} \mathbf{BLoc}_0$ and a geometric f-functor $\mathscr{F}$ from a small_2 $\mathbf{X}_+$-topos $\mathscr{C}_+$ to a small_2 $\mathbf{X}_-$-topos $\mathscr{C}_-$.

(7.2.8) Its morphisms from $(\mathbf{f}: \mathbf{X}_- \to \mathbf{X}_+, \mathscr{F}: \mathscr{C}_+ \to \mathscr{C}_-)$ to $(\mathbf{f}': \mathbf{X}'_- \to \mathbf{X}'_+, \mathscr{F}': \mathscr{C}'_+ \to \mathscr{C}'_-)$ are all triples

$$((\mathbf{h}_-: \mathbf{X}_- \to \mathbf{X}'_-,\mathscr{H}_-: \mathscr{C}'_- \to \mathscr{C}_-), (\mathbf{h}_+: \mathbf{X}_+ \to \mathbf{X}'_+,\mathscr{H}_+: \mathscr{C}'_+ \to \mathscr{C}_+), \alpha)$$

where

$$(7.2.8.1)\ (\mathbf{h}_\pm: \mathbf{X}_\pm \to \mathbf{X}'_\pm,\mathscr{H}_\pm: \mathscr{C}'_\pm \to \mathscr{C}_\pm) \in \operatorname{Ob} \mathbf{BGEOM}_2^*$$

(7.2.8.2) $h_+ \circ f = f' \circ h_-$

(7.2.8.3) α is a natural $h_+ \circ f$-transformation from $\mathscr{H}_- \circ \mathscr{F}'$ to $\mathscr{F} \circ \mathscr{H}_+$.

(7.2.9) Its 2-arrows from $((h_-, \mathscr{H}_-), (h_+, \mathscr{H}_+), \alpha): (f, \mathscr{F}) \to (f', \mathscr{F}')$ to $((k_-, \mathscr{K}_-), (k_+, \mathscr{K}_+), \beta): (f, \mathscr{F}) \to (f', \mathscr{F}')$ exist provided that

(7.2.9.1) $h_\pm = k_\pm$

and are all pairs (σ_-, σ_+), where

(7.2.9.2) $\sigma_\pm$ are natural $h_\pm$-transformations from $\mathscr{H}_\pm$ to $\mathscr{K}_\pm$.

(7.2.9.3) $\beta \cdot (\sigma_- \circ \mathscr{F}') = (\mathscr{F} \circ \sigma_+) \cdot \alpha$

By the same token as in Section 2.1, we have the following result.

Theorem 7.2.2. $\mathbf{BGEOM}^2_*$ is a 2-category with respect to the following operations:

(7.2.10) Given two morphisms

$$((h_-, \mathscr{H}_-), (h_+, \mathscr{H}_+), \alpha): (f, \mathscr{F}) \to (f', \mathscr{F}')$$

$$((k_-, \mathscr{K}_-), (k_+, \mathscr{K}_+), \beta): (f', \mathscr{F}') \to (f'', \mathscr{F}'')$$

then

$$((k_-, \mathscr{K}_-), (k_+, \mathscr{K}_+), \beta) \circ ((h_-, \mathscr{H}_-), (h_+, \mathscr{H}_+), \alpha)$$

shall be

$$((k_- \circ h_-, \mathscr{H}_- \circ \mathscr{K}_-), (k_+ \circ h_+, \mathscr{H}_+ \circ \mathscr{K}_+), (\alpha \circ \mathscr{K}_+) \cdot (\mathscr{H}_- \circ \beta))$$

(7.2.11) Given three parallel morphisms $((h_-, \mathscr{H}_-), (h_+, \mathscr{H}_+), \alpha)$, $((k_-, \mathscr{K}_-), (k_+, \mathscr{K}_+), \beta)$, and $((l_-, \mathscr{L}_-), (l_+, \mathscr{L}_+), \gamma): (f, \mathscr{F}) \to (f', \mathscr{F}')$, the vertical composite of 2-arrows

$$(\sigma_-, \sigma_+): ((h_-, \mathscr{H}_-), (h_+, \mathscr{H}_+), \alpha) \to ((k_-, \mathscr{K}_-), (k_+, \mathscr{K}_+), \beta)$$

$$(\tau_-, \tau_+): ((k_-, \mathscr{K}_-), (k_+, \mathscr{K}_+), \beta) \to ((l_-, \mathscr{L}_-), (l_+, \mathscr{L}_+), \gamma)$$

in notation $(\tau_-, \tau_+) \cdot (\sigma_-, \sigma_+)$, shall be $(\tau_- \cdot \sigma_-, \tau_+ \cdot \sigma_+)$.

(7.2.12) Given two pairs of parallel morphisms

$$((h_-, \mathscr{H}_-), (h_+, \mathscr{H}_+), \alpha), ((k_-, \mathscr{K}_-), (k_+, \mathscr{K}_+), \beta): (f, \mathscr{F}) \to (f', \mathscr{F}')$$

$$((h'_-, \mathscr{H}'_-), (h'_+, \mathscr{H}'_+), \alpha'), ((k'_-, \mathscr{K}'_-), (k_+, \mathscr{K}'_+), \beta'): (f', \mathscr{F}') \to (f'', \mathscr{F}'')$$

the horizontal composite of 2-arrows

$$(\sigma_-, \sigma_+): ((h_-, \mathscr{H}_-), (h_+, \mathscr{H}_+), \alpha) \to ((k_-, \mathscr{K}_-), (k_+, \mathscr{K}_-), \beta)$$

$$(\tau_-, \tau_+): ((h'_-, \mathscr{H}'_-), (h'_+, \mathscr{H}'_+), \alpha') \to ((k'_-, \mathscr{K}'_-), (k'_+, \mathscr{K}'_+), \beta')$$

in notation $(\tau_-, \tau_+) \circ (\sigma_-, \sigma_+)$, shall be $(\sigma_- \circ \tau_-, \sigma_+ \circ \tau_+)$.

The objects and morphisms of $\mathbf{BGEOM}_2^*$ constitute a category which we denote by $\mathbf{BGeom}_2^*$.

We close this subsection with a proposition connecting the two categories $\mathbf{BGeom}_2^*$ and $\mathbf{BGeom}_*^2$, for which we first need to fix some notation and terminology. Let us consider the following diagram in $\mathbf{BTop}_2$, in which it is assumed that $e \circ h_- = h_+ \circ f$ and $f \circ k_- = k_+ \circ g$:

$$
\begin{array}{ccc}
(X_+, \mathscr{S}_+) & \xleftarrow{(e, \mathscr{E})} & (X_-, \mathscr{S}_-) \\
\uparrow{\scriptstyle (h_+, \mathscr{H}_+)} & & \uparrow{\scriptstyle (h_-, \mathscr{H}_-)} \\
(Y_+, \mathscr{T}_+) & \xleftarrow{(f, \mathscr{F})} & (Y_-, \mathscr{T}_-) \\
\uparrow{\scriptstyle (k_+, \mathscr{K}_+)} & & \uparrow{\scriptstyle (k_-, \mathscr{K}_-)} \\
(Z_+, \mathscr{U}_+) & \xleftarrow{(g, \mathscr{G})} & (Z_-, \mathscr{U}_-)
\end{array}
$$

Given a natural $e \circ h_-$-transformation $\alpha: \mathscr{F} \circ \mathscr{H}_+ \to \mathscr{H}_- \circ \mathscr{E}$ and a natural $f \circ k_-$-transformation $\beta: \mathscr{G} \circ \mathscr{K}_+ \to \mathscr{K}_- \circ \mathscr{F}$, we denote the third component of the composite $((h_-, \mathscr{H}_-), (h_+, \mathscr{H}_+), \beta) \circ ((k_-, \mathscr{K}_-), (k_+, \mathscr{K}_+), \alpha)$ in the category $\mathbf{BGeom}_2^*$ by $\beta \circ {}^*\alpha$, which is surely a natural $e \circ h_- \circ k_-$-transformation from $\mathscr{G} \circ \mathscr{K}_+ \circ \mathscr{H}_+$ to $\mathscr{K}_- \circ \mathscr{H}_- \circ \mathscr{E}$.

Let us consider the following diagram in $\mathbf{BTop}_2$, in which it is assumed that $e \circ h_- = h_+ \circ f$ and $f \circ k_- = k_+ \circ g$:

$$
\begin{array}{ccccc}
(X_+, \mathscr{S}_-) & \xleftarrow{(h_+, \mathscr{H}_+)} & (Y_+, \mathscr{T}_+) & \xleftarrow{(k_+, \mathscr{K}_+)} & (Z_+, \mathscr{U}_+) \\
{\scriptstyle (e, \mathscr{E})}\uparrow & & {\scriptstyle (f, \mathscr{F})}\uparrow & & {\scriptstyle (g, \mathscr{G})}\uparrow \\
(X_-, \mathscr{S}_-) & \xleftarrow[(h_-, \mathscr{H}_-)]{} & (Y_-, \mathscr{T}_-) & \xleftarrow[(k_-, \mathscr{K}_-)]{} & (Z_-, \mathscr{U}_-)
\end{array}
$$

Given a natural $e \circ h_-$-transformation $\alpha: \mathscr{H}_- \circ \mathscr{E} \to \mathscr{F} \circ \mathscr{H}_+$ and a natural $f \circ k_-$-transformation $\beta: \mathscr{K}_- \circ \mathscr{F} \to \mathscr{G} \circ \mathscr{K}_+$, we denote the third component of the composite $((h_-, \mathscr{H}_-), (h_+, \mathscr{H}_+), \alpha) \circ ((k_-, \mathscr{K}_-), (k_+, \mathscr{K}_+), \beta)$ within the category $\mathbf{BGeom}_*^2$ by $\beta \circ {}_*\alpha$, which is surely a natural $h_+ \circ k_+ \circ g$-transformation from $\mathscr{K}_- \circ \mathscr{H}_- \circ \mathscr{E}$ to $\mathscr{G} \circ \mathscr{K}_+ \circ \mathscr{H}_+$.

By the same token as in Proposition 2.1.7, we have:

Proposition 7.2.3. Consider the following diagram within the category $\mathbf{BTop}_2$, in which it is assumed that $d_1 \circ e_2 = e_1 \circ d_2$, $g_1 \circ e_3 = e_2 \circ g_2$, $d_2 \circ f_2 = f_1 \circ d_3$, and $g_2 \circ f_3 = f_2 \circ g_3$.

$$
\begin{array}{ccccc}
(X_1, \mathscr{S}_1) & \xleftarrow{(e_1, \mathscr{E}_1)} & (X_2, \mathscr{S}_2) & \xleftarrow{(f_1, \mathscr{F}_1)} & (X_3, \mathscr{S}_3) \\
(d_1, \mathscr{D}_1) \uparrow & & (d_2, \mathscr{D}_2) \uparrow & & (d_3, \mathscr{D}_3) \uparrow \\
(Y_1, \mathscr{T}_1) & \xleftarrow{(e_2, \mathscr{E}_2)} & (Y_2, \mathscr{T}_2) & \xleftarrow{(f_2, \mathscr{F}_2)} & (Y_3, \mathscr{T}_3) \\
(g_1, \mathscr{G}_1) \uparrow & & (g_2, \mathscr{G}_2) \uparrow & & (g_3, \mathscr{G}_3) \uparrow \\
(Z_1, \mathscr{U}_1) & \xleftarrow{(e_3, \mathscr{E}_3)} & (Z_2, \mathscr{U}_2) & \xleftarrow{(f_3, \mathscr{F}_3)} & (Z_3, \mathscr{U}_3)
\end{array}
$$

Given a natural $d_1 \circ e_2$-transformation $\alpha: \mathscr{E}_2 \circ \mathscr{D}_1 \to \mathscr{D}_2 \circ \mathscr{E}_1$, a natural $g_1 \circ e_3$-transformation $\alpha': \mathscr{E}_3 \circ \mathscr{G}_1 \to \mathscr{G}_2 \circ \mathscr{E}_2$, a natural $d_2 \circ f_2$-transformation β: $\mathscr{F}_2 \circ \mathscr{D}_2 \to \mathscr{D}_3 \circ \mathscr{F}_1$, and a natural $g_2 \circ f_3$-transformation $\beta': \mathscr{F}_3 \circ \mathscr{G}_2 \to \mathscr{G}_3 \circ \mathscr{F}_2$, we have

$$(\beta' \circ {}^*\beta) \circ {}_*(\alpha' \circ {}^*\alpha) = (\beta' \circ {}_*\alpha') \circ {}^*(\beta \circ {}_*\alpha)$$

7.3. The 2-Category $\mathbf{ETOP_2}$

Let $\mathscr{M}$ be a manual of Boolean locales, which shall be fixed throughout this subsection. The 2-category $\mathbf{ETOP_2}(\mathscr{M})$ or $\mathbf{ETOP_2}$ for short consists of the following entities:

(7.3.1) Its objects are all small_2 $\mathscr{M}$-toposes.

(7.3.2) Given two small_2 $\mathscr{M}$-toposes $\mathfrak{G}$ and $\mathfrak{T}$, its morphisms from $\mathfrak{G}$ and $\mathfrak{T}$ are all pairs $(\mathfrak{H}, \alpha)$ where:

- (7.3.2.1) $\mathfrak{H}$ is an assignment, to each $X \in \mathrm{Ob}\, \mathscr{M}$, of a geometric X-functor $\mathfrak{H}(X)$ from $\mathfrak{G}(X)$ to $\mathfrak{T}(X)$.
- (7.3.2.2) α is an assignment, to each $f: X \to Y \in \mathrm{Mor}\, \mathscr{M}$, of a natural f-transformation $\alpha(f)$ from $\mathfrak{T}(f) \circ \mathfrak{H}(Y)$ to $\mathfrak{F}(X) \circ \mathfrak{G}(f)$, so that $((f, \mathfrak{T}(f)), (f, \mathfrak{G}(f)), \alpha(f))$ is a morphism from $(id_X, \mathfrak{H}(X))$ to $(id_Y, \mathfrak{H}(Y))$ in the 2-category $\mathbf{BGEOM}^2_*$ on the one hand, and $((id_X, \mathfrak{H}(X)), (id_Y, \mathfrak{H}(Y)), \alpha(f))$ is a morphism from $(f, \mathfrak{T}(f))$ to $(f, \mathfrak{G}(f))$ in the 2-category $\mathbf{BGEOM}^*_2$ on the other hand.
- (7.3.2.3) The assignments $X \in \mathrm{Ob}\, \mathscr{M} \mapsto (id_X, \mathfrak{H}(X))$ and $f \in \mathrm{Mor}\, \mathscr{M} \mapsto ((f, \mathfrak{T}(f)), (f, \mathfrak{G}(f)), \alpha(f))$ constitute a functor from the category $\mathscr{M}$ to the category $\mathbf{BGeom}^2_*$. In particular, for any $f: X \to Y \in \mathrm{Mor}\, \mathscr{M}$ and any $g: Y \to Z \in \mathrm{Mor}\, \mathscr{M}$, we have $\alpha(g \circ f) = (\alpha(f) \circ \mathfrak{G}(g)) \cdot (\mathfrak{T}(f) \circ \alpha(g))$.

(7.3.3) Its 2-arrows from $(\mathfrak{H}, \alpha)$: $\mathfrak{G} \to \mathfrak{T}$ to $(\mathfrak{K}, \beta)$: $\mathfrak{G} \to \mathfrak{T}$ are all assignments σ, to each $X \in \text{Ob}\,\mathcal{M}$, of a natural X-transformation $\sigma(X)$: $\mathfrak{H}(X) \to \mathfrak{K}(X)$ such that for any f: $X \to Y \in \text{Mor}\,\mathcal{M}$, $(\sigma(X), \sigma(Y))$ is a 2-arrow from

$$((\text{id}_X, \mathfrak{H}(X)), (\text{id}_Y, \mathfrak{H}(Y)), \alpha(f)): (f: \mathfrak{T}(f)) \to (f, \mathfrak{G}(f))$$

to

$$((\text{id}_X, \mathfrak{K}(X)), (\text{id}_Y, \mathfrak{K}(Y)), \beta(f)): (f, \mathfrak{T}(f)) \to (f, \mathfrak{G}(f))$$

within the 2-category $\mathbf{BGEOM}_2^*$.

Theorem 7.3.1. $\mathbf{ETOP}_2$ is a 2-category with respect to the following operations:

(7.3.4) Given three small$_2$ $\mathcal{M}$-toposes $\mathfrak{G}$, $\mathfrak{T}$, and $\mathfrak{U}$, the composite $(\mathfrak{L}, \gamma)$ of $(\mathfrak{H}, \alpha)$: $\mathfrak{G} \to \mathfrak{T}$ and $(\mathfrak{K}, \beta)$: $\mathfrak{T} \to \mathfrak{U}$, in notation $(\mathfrak{L}, \gamma) = (\mathfrak{K}, \beta) \circ (\mathfrak{H}, \alpha)$, shall be such that:

(7.3.4.1) For any $X \in \text{Ob}\,\mathcal{M}$, $\mathfrak{L}(X) = \mathfrak{K}(X) \circ \mathfrak{H}(X)$.
(7.3.4.2) For any f: $X \to Y \in \text{Mor}\,\mathcal{M}$, we have

$$((\text{id}_X\, \mathfrak{L}(X)), (\text{id}_Y, \mathfrak{L}(Y)), \gamma(f))$$
$$= ((\text{id}_X, \mathfrak{H}(X)), (\text{id}_Y, \mathfrak{H}(Y)), \alpha(f))$$
$$\circ\, ((\text{id}_X, \mathfrak{K}(X)), (\text{id}_Y, \mathfrak{K}(Y)), \beta(f))$$

within the 2-category $\mathbf{BGEOM}_2^*$.

(7.3.5) Given three morphisms $(\mathfrak{H}, \alpha)$, $(\mathfrak{K}, \beta)$, and $(\mathfrak{L}, \gamma)$ from a small$_2$ $\mathcal{M}$-topos $\mathfrak{G}$ to a small$_2$ $\mathcal{M}$-topos $\mathfrak{T}$, the vertical composite ν of 2-arrows σ: $(\mathfrak{H}, \alpha) \to (\mathfrak{K}, \beta)$ and τ: $(\mathfrak{K}, \beta) \to (\mathfrak{L}, \gamma)$, in notation $\nu = \tau \cdot \sigma$, shall be such that for any f: $X \to Y \in \text{Mor}\,\mathcal{M}$, we have

$$(\nu(X), \nu(Y)) = (\tau(X), \tau(Y)) \cdot (\sigma(X), \sigma(Y))$$

within the 2-catêgory $\mathbf{BGEOM}_2^*$.

(7.3.6) Given two pairs of parallel morphisms $(\mathfrak{H}, \alpha)$, $(\mathfrak{K}, \beta)$: $\mathfrak{G} \to \mathfrak{T}$ and $(\mathfrak{H}', \alpha')$, $(\mathfrak{K}', \beta')$: $\mathfrak{T} \to \mathfrak{U}$, the horizontal composite ν of 2-arrows σ: $(\mathfrak{H}, \alpha) \to (\mathfrak{K}, \beta)$ and τ: $(\mathfrak{H}', \beta') \to (\mathfrak{K}', \beta')$, in notation $\nu = \tau \circ \sigma$, shall be such that for any f: $X \to Y \in \text{Mor}\,\mathcal{M}$, we have

$$(\nu(X), \nu(Y)) = (\sigma(X), \sigma(Y)) \circ (\tau(X), \tau(Y))$$

within the 2-category $\mathbf{BGEOM}_2^*$.

Outline of the proof. The 2-categorical structure of $\mathbf{ETOP}_2$ follows largely from that of $\mathbf{BGEOM}_2^*$. It remains only to note that γ in (7.3.4) is guaranteed to satisfy the condition (7.3.2.3) by dint of Proposition 7.2.3. ∎

8. EMPIRICAL CLASSIFYING TOPOSES

8.1. Rings Within Boolean Toposes

Let X be a Boolean locale and $\mathcal{T}$ a small_2 X-topos. By interpreting the notion of **Rng**($\mathcal{T}$) within the topos $\mathbf{BEns}_2$(X), we gt the notion of $\mathcal{BRng}$(X; $\mathcal{T}$) externally. Similarly, by interpreting the notions of **LocRng**($\mathcal{T}$) and **fp-Rng**($\mathcal{T}$) within the topos $\mathbf{BEns}_2$(X), we get the notions of $\mathcal{BLocRng}$(X; $\mathcal{T}$) and $\mathcal{fp}$-$\mathcal{BRng}$(X; $\mathcal{T}$) respectively. We write **BRng**(X; $\mathcal{T}$), **BLocRng**(X; $\mathcal{T}$), and **fp-BRng**(X; $\mathcal{T}$) for $\mathcal{BRng}$(X; $\mathcal{T}$)[1_X], $\mathcal{BLocRng}$(X; $\mathcal{T}$)[1_X], and $\mathcal{fp}$-$\mathcal{BRng}$(X; $\mathcal{T}$)[1_X], respectively. If the topos $\mathcal{T}$ is $\mathcal{BEns}_i$(X) (i = 0, 1), then $\mathcal{BRng}$(X; $\mathcal{T}$), $\mathcal{BLocRng}$(X; $\mathcal{T}$), $\mathcal{fp}$-$\mathcal{BRng}$(X; $\mathcal{T}$), **BRng**(X; $\mathcal{T}$), **BLocRng**(X; $\mathcal{T}$) and **fp-BRng**(X; $\mathcal{T}$) are denoted by $\mathcal{BRng}_i$(X), $\mathcal{BLocRng}_i$(X), $\mathcal{fp}$-$\mathcal{BRng}_i$(X), $\mathbf{BRng}_i$(X), $\mathbf{BLocRng}_i$(X), and $\mathbf{fp\text{-}BRng}_i$(X), respectively. We denote by $\mathcal{L}_{\text{Zar}}$[X] of $\mathcal{L}_{\text{Zar}}$ the Grothendieck X-topology on $\mathcal{fp}$-$\mathcal{BRng}_0^{\text{op}}$ obtained simply by interpreting $\mathbf{L}_{\text{Zar}}$ within the topos $\mathbf{BEns}_2$(X).

Let f: $X_- \to X_+$ be a morphism of Boolean locales and $\mathcal{F}: \mathcal{T}_+ \to \mathcal{T}_-$ a geometric f-functor of small_2 $X_\pm$-toposes. Then $\mathcal{F}$ naturally induces f-functors $\mathcal{F}_{\mathcal{BRng}}: \mathcal{BRng}(X_+; \mathcal{T}_+) \to \mathcal{BRng}(X_-; \mathcal{F}_-)$ and $\mathcal{F}_{\mathcal{BLocRng}}: \mathcal{BLocRng}(X_+; \mathcal{T}_+) \to \mathcal{BLocRng}(X_-; \mathcal{F}_-)$. The geometric f-functor $f^*_{\mathcal{BEns}_0}: \mathcal{BEns}_0(X_+) \to \mathcal{BEns}_0(X_-)$ naturally induces an f-morphism

$$\mathcal{F}_{\mathcal{fp}\text{-}\mathcal{BRng}}\text{op}: \quad (\mathcal{fp}\text{-}\mathcal{BRng}(X_+; \mathcal{T}_+)^{\text{op}}, \mathcal{L}_{\text{Zar}}) \to (\mathcal{fp}\text{-}\mathcal{BRng}(X_-; \mathcal{F}_-)^{\text{op}}, \mathcal{L}_{\text{Zar}})$$

of $X_\pm$-sites.

8.2. Rings Within Empirical Toposes

Let $\mathcal{M}$ be a manual of Boolean locales, which shall be fixed throughout this subsection. Let $\mathfrak{T}$ be a small_2 $\mathcal{M}$-topos. Its associated $\mathcal{M}$-category $\mathfrak{ERng}(\mathcal{M}; \mathfrak{T})$ is defined as follows:

(8.2.1) For each X ∈ Ob $\mathcal{M}$, $\mathfrak{ERng}(\mathcal{M}; \mathfrak{T})$(X) shall be $\mathcal{Rng}$(X; $\mathfrak{T}$(X)).
(8.2.2) For each f: X → Y ∈ Mor $\mathcal{M}$, $\mathfrak{ERng}(\mathcal{M}; \mathfrak{T})$(f) shall be

$$\mathfrak{T}(f)_{\mathcal{BRng}}: \quad \mathcal{BRng}(Y; \mathfrak{T}(Y)) \to \mathcal{BRng}(X; \mathfrak{T}(X))$$

We write **ERng**($\mathfrak{T}$) for **EObj**($\mathfrak{ERng}(\mathcal{M}; \mathfrak{T})$).

Given a small_2 $\mathcal{M}$-topos $\mathfrak{T}$, we define its associated $\mathcal{M}$-category $\mathfrak{ELocRng}(\mathcal{M}; \mathfrak{T})$ as follows:

(8.2.3) For each X ∈ Ob $\mathcal{M}$, $\mathfrak{ELocRng}(\mathcal{M}; \mathfrak{T})$(X) shall be $\mathcal{BLocRng}$(X; $\mathfrak{T}$(X)).
(8.2.4) For each f: X → Y ∈ Mor $\mathcal{M}$, $\mathfrak{ELocRng}(\mathcal{M}; \mathfrak{T})$(f) shall be

$$\mathfrak{T}(f)_{\mathcal{BLocRng}}: \quad \mathcal{BLocRng}(Y; \mathfrak{T}(Y)) \to \mathcal{BLocRng}(X; \mathfrak{T}(X))$$

We write **ELocRng**($\mathfrak{T}$) for **EObj**($\mathfrak{ELocRng}(\mathcal{M}; \mathfrak{T})$).

The $\mathcal{M}$-finitely $\mathcal{M}$-cocomplete $\mathcal{M}$-category $\mathfrak{fp}\text{-}\mathfrak{C}\mathfrak{R}\mathfrak{ng}_0(\mathcal{M})$ is defined as follows:

(8.2.5) For each $X \in \mathrm{Ob}\ \mathcal{M}$, $\mathfrak{fp}\text{-}\mathfrak{C}\mathfrak{R}\mathfrak{ng}_0(\mathcal{M})(X)$ shall be $\mathfrak{fp}\text{-}\mathscr{R}ng_0(X)$.

(8.2.6) For each $f: X \to Y \in \mathrm{Mor}\ \mathcal{M}$, $\mathfrak{fp}\text{-}\mathfrak{C}\mathfrak{R}\mathfrak{ng}_0(\mathcal{M})(f)$ shall be $f^*_{fp\text{-}\mathscr{BR}ng_0}$: $fp\text{-}\mathscr{BR}ng(Y) \to fp\text{-}\mathscr{BR}ng_0(X)$

The assignment $X \in \mathrm{Ob}\ \mathcal{M} \mapsto \mathscr{L}_{\mathrm{Zar}}[X]$ gives a Grothendieck $\mathcal{M}$-topology on $\mathfrak{fp}\text{-}\mathfrak{C}\mathfrak{R}\mathfrak{ng}_0(\mathcal{M})^{\mathrm{op}}$.

8.3. The First Preliminary Theorems

Let X be a Boolean locale, which shall be fixed throughout this subsection. We denote by $\mathbf{BTOP}_2[X]$ the sub-2-category of $\mathbf{BTOP}_2$ whose objects are all pairs $(X;\mathscr{T})$ of X and a small$_2$ X-topos $\mathscr{T}$, whose morphisms from $(X, \mathscr{S})$ to $(X, \mathscr{T})$ are all pairs $(\mathrm{id}_X, \mathscr{F})$ of id_X and a geometric X-functor from $\mathscr{T}$ to $\mathscr{S}$, and whose 2-arrows from $(\mathrm{id}_X, \mathscr{F})$: $(X, \mathscr{S}) \to (X, \mathscr{T})$ to $(\mathrm{id}_X, \mathscr{G})$: $(X, \mathscr{S}) \to (X, \mathscr{T})$ are all natural X-transformations from $\mathscr{F}$ to $\mathscr{G}$. For simplicity we will identify $(X, \mathscr{T})$ with $\mathscr{T}$ and $(\mathrm{id}_X, \mathscr{F})$ with $\mathscr{F}$ within the 2-category $\mathbf{BTOP}_2\{X]$, but the reader should not forget that the direction of a geometric X-functor as a morphism of $\mathbf{BTOP}_2[X]$ is the reverse of that of the geometric X-functor itself.

We denote by $\mathbf{BTop}_2[X]$ the category that the objects and the morphisms of $\mathbf{BTop}_2[X]$ constitute. Given a small$_2$ X-topos $\mathscr{S}$, the assignment $\mathscr{T} \in \mathrm{Ob}\ \mathbf{BTop}_2[X] \mapsto \mathbf{BTOP}_2[X](\mathscr{T},\mathscr{S})$ naturally induces a contravariant functor from $\mathbf{BTop}_2[X]$ to $\mathbf{Cat}_2$, while the assignments $\mathscr{T} \in \mathrm{Ob}\ \mathbf{BTop}_2[X] \mapsto \mathbf{BRng}(X;\mathscr{T})$ and $\mathscr{T} \in \mathrm{Ob}\ \mathbf{BTop}_2[\downarrow X] \mapsto \mathbf{BLocRng}(\mathscr{T})$ naturally give rise to contravariant functors from $\mathbf{BTop}_2[X]$ to $\mathbf{Cat}_2$.

By simply Booleanizing Theorems 2.4.1 and 2.4.2, we get the following theorems:

Theorem 8.3.1. For any small$_1$-X-cocomplete small$_2$ X-topos $\mathscr{T}$, there is an equivalence of categories

$$\mathbf{BTOP}_2[X](\mathscr{T}, \mathscr{BP}re\mathscr{S}h_1(X; fp\text{-}\mathscr{BR}ng_0^{\mathrm{op}}(X))) \xrightarrow{\sim} \mathbf{BRng}(\mathscr{T})$$

which is natural in $\mathscr{T}$.

Theorem 8.3.2. For any small$_1$-X-cocomplete small$_2$ X-topos $\mathscr{T}$, there is an equivalence of categories

$$\mathbf{BTOP}_2[X](\mathscr{T}, \mathscr{BS}h_1(X; fp\text{-}\mathscr{BR}ng_0^{\mathrm{op}}(X), \mathscr{L}_{\mathrm{Zar}})) \xrightarrow{\sim} \mathbf{BLocRng}(\mathscr{T})$$

which is natural in $\mathscr{T}$.

8.4. The Second Preliminary Theorems

Given $f: Y \to Z \in$ Mor $\mathbf{BLoc}_0$, a small$_2$ Y-topos $\mathscr{S}$, and a small$_2$ Z-topos $\mathscr{T}$, we denote by $\mathbf{BTOP}_2[f]((Y, \mathscr{S}), (Z, \mathscr{T}))$ the full subcategory of $\mathbf{BTOP}_2((Y, \mathscr{S}), (Z, \mathscr{T}))$ whose objects are all $(g: Y \to Z, \mathscr{G}: \mathscr{T} \to \mathscr{S}) \in \mathbf{BTOP}_2((Y, \mathscr{S}), (Z, \mathscr{T}))$ with $g = f$.

We now introduce a category to be denoted by $\mathbf{BTop}_2[\downarrow]$. Its objects are all pairs $(f: X_- \to X_+, \mathscr{T}_-)$ of $f \in$ Mor $\mathbf{BLoc}_0$ and a small$_2$ X_--topos $\mathscr{T}_-$. Its morphisms from $(f: X_- \to X_+, \mathscr{T}_-)$ to $(f': X'_- \to X'_+, \mathscr{T}'_-)$ are all triples $(h_-, h_+, \mathscr{H}_-)$ subject to the following conditions:

(8.4.1) $(h_-, \mathscr{H}_-): (X_-, \mathscr{T}_-) \to (X'_-, \mathscr{T}'_-) \in$ Mor $\mathbf{BTop}_2$
(8.4.2) $h_+: X'_+ \to X_+ \in$ Mor $\mathbf{BLoc}_0$
(8.4.3) $h_+ \circ f' \circ h_- = f$

The composition of

$$(h_-, h_+, \mathscr{H}_-): \quad (f: X_- \to X_+, \mathscr{T}_-) \to (f': X'_- \to X'_+, \mathscr{T}'_-)$$

and

$$(k_-, k_+, \mathscr{K}_-): \quad (f': X'_- \to X'_+, \mathscr{T}'_-) \to (f'': X''_- \to X''_+, \mathscr{T}''_-)$$

within the category $\mathbf{BTop}_2[\downarrow]$, to be denoted by $(k_-, k_+, \mathscr{K}_-) \circ (h_-, h_+, \mathscr{H}_-)$, is defined to be $(k_- \circ h_-, h_+ \circ k_+, \mathscr{H}_- \circ \mathscr{K}_-)$.

We denote by $\mathbf{BTop}_2[\downarrow X]$ the category whose objects are all triples $(Y, \mathscr{T}, f)$ with $(Y, \mathscr{T}) \in$ Ob $\mathbf{BTop}_2$ and $f: Y \to X \in$ Mor $\mathbf{BLoc}_0$ and whose morphisms from $(Y, \mathscr{T}, f)$ to $(Z, \mathscr{S}, g)$ are all $(h, \mathscr{H}): (Y, \mathscr{T}) \to (Z, \mathscr{S}) \in$ Mor $\mathbf{BTop}_2$ with $g \circ h = f$. The category $\mathbf{BTop}_2[\downarrow X]$ can naturally be regarded as a subcategory of $\mathbf{BTop}_2[\downarrow]$ whose objects are all $(f: X_- \to X_+, \mathscr{T}_-) \in$ Ob $\mathbf{BTop}_2[\downarrow]$ with $X_+ = X$ and whose morphisms are all $(h_-, h_+, \mathscr{H}_-) \in$ Mor $\mathbf{BTop}_2[\downarrow]$ with $h_+ = id_X$. Given a small$_2$ X-topos $\mathscr{S}$, the assignment $(Y, \mathscr{T}, f) \in$ Ob $\mathbf{BTOP}_2[\downarrow X] \mapsto \mathbf{BTOP}_2[f]((Y, \mathscr{T}), (X, \mathscr{S}))$ naturally induces a contravariant functor from $\mathbf{BTop}_2[\downarrow X]$ to $\mathbf{Cat}_2$, while the assignments $(Y, \mathscr{T}, f) \in$ Ob $\mathbf{BTop}_2[\downarrow X] \mapsto \mathbf{BRng}(\mathscr{T})$ and $(Y, \mathscr{T}, f) \in$ Ob $\mathbf{BTop}_2[\downarrow X] \mapsto \mathbf{BLocRng}(\mathscr{T})$ naturally give rise to contravariant functors from $\mathbf{BTop}_2[\downarrow X]$ to $\mathbf{Cat}_2$.

Theorems 8.3.1 and 8.3.2 are generalized as follows:

Theorem 8.4.1. For any $(Y, \mathscr{T}, f) \in$ Ob $\mathbf{BTop}_2[\downarrow X]$ with $\mathscr{T}$ being small$_1$-Y-cocomplete, there is an equivalence of categories

$$\mathbf{BTOP}_2[f]((Y, \mathscr{T}), (X, \mathscr{BPreSh}_1(X; fp\text{-}\mathscr{BRng}_0^{op}(X)))) \tilde{\to} BRng(\mathscr{T})$$

which is natural in $(Y, \mathscr{T}, f)$.

Theorem 8.4.2. For any $(Y, \mathscr{T}, f) \in$ Ob $\mathbf{BTOP}_2[\downarrow X]$ with $\mathscr{T}$ being small$_1$-Y-cocomplete, there is an equivalence of categories

$$\mathbf{BTOP}_2[f]((Y, \mathscr{T}), (X, \mathscr{BSh}_1(X; \mathscr{fp}\text{-}\mathscr{BRng}_0^{op}(X), \mathscr{L}_{Zar}))) \xrightarrow{\sim} \mathbf{BLocRng}(\mathscr{T})$$

which is natural in $(Y, \mathscr{T}, f)$.

8.5. The Main Theorems

Let $\mathscr{M}$ be a manual of Boolean locales, which shall be fixed throughout this subsection. By Theorem 8.4.1 it is easy to see the following.

Theorem 8.5.1. For any small$_1$-cocomplete small$_2$ $\mathscr{M}$-topos $\mathfrak{T}$, there is an equivalence of categories

$$\mathbf{ETOP}_2(\mathfrak{EBreSh}_1(\mathscr{M}; \mathfrak{fp}\text{-}\mathfrak{ERng}_0^{op}(\mathscr{M})), \mathfrak{T}) \xrightarrow{\sim} \mathbf{ERng}(\mathfrak{T})$$

which is natural in $\mathfrak{T}$.

By Theorem 8.4.2 we can see easily the following:

Theorem 8.5.2. For any small$_1$-cocomplete small$_2$ $\mathscr{M}$-topos $\mathfrak{T}$, there is an equivalence of categories

$$\mathbf{ETOP}_2(\mathfrak{ESh}_1(\mathscr{M}; \mathfrak{fp}\text{-}\mathfrak{ERng}_0^{op}(\mathscr{M}), \mathfrak{L}_{Zar}), \mathfrak{T}) \xrightarrow{\sim} \mathbf{ELocRng}(\mathfrak{T})$$

which is natural in $\mathfrak{T}$.

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